Millimeter interferometers

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Millimeter interferometers

Outline

- Goal: the van Cittert–Zernike theorem
- The ideal interferometer
  - $\hookrightarrow$ geometrical delay, source size, bandwidth
- The real interferometer
  - $\hookrightarrow$ heterodyne receivers, delay correction, correlators
- Aperture synthesis
  - $\hookrightarrow$ uv plane, field of view, transfer function
- Sensitivity

Goal
van Cittert–Zernike theorem

- Wiener-Kichnine theorem
  - autocorrelation of $S(t)$ = FT(spectra)
    $S(t_1)\,S(t_2) = AS(\tau) \rightleftharpoons S(\nu)$
  - implementation: FT spectrometers
- **van Cittert–Zernike theorem**
  - spatial autocorrelation of $S(x)$ = FT(brightness)
    $S(x_1)\,S(x_2) = AS(u) \rightleftharpoons S(\alpha)$
  - implementation: interferometry

Implementing the van Cittert–Zernike theorem

1. Build a device that measures the spatial autocorrelation of the incoming signal
2. Do it for all possible scales
3. Take the FT and get an image of the brightness distribution

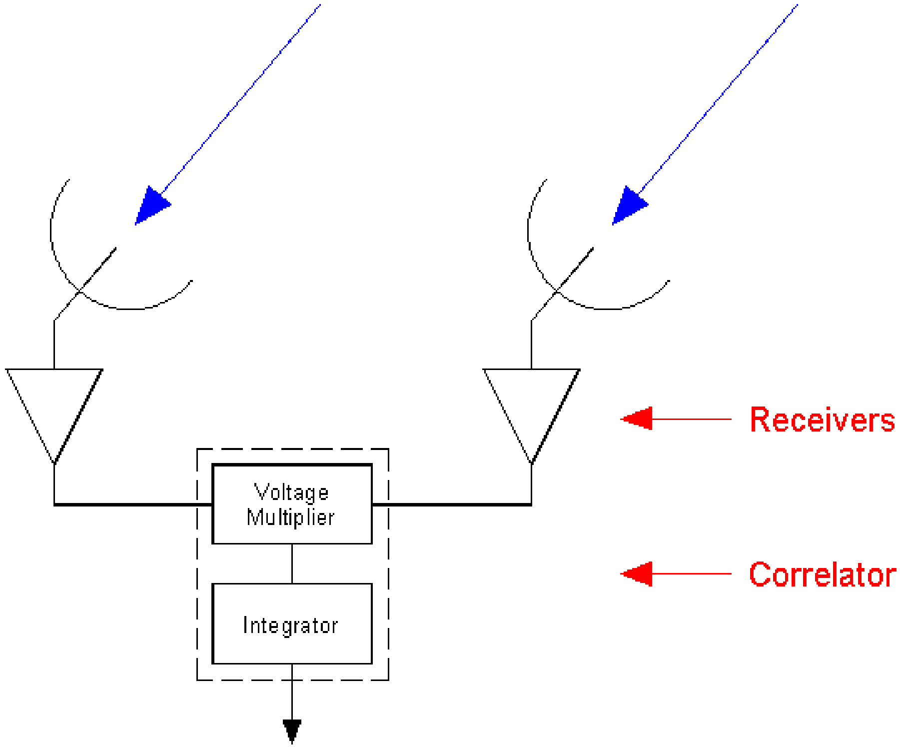
Interferometry

Implementing the van Cittert–Zernike theorem

1. Build a device that measures the spatial autocorrelation of the incoming signal $\longrightarrow$ **2-elements interferometer**
2. Do it for all possible scales $\longrightarrow$ **N antennas**
3. Take the FT and get an image of the brightness distribution $\longrightarrow$ **software**

The ideal interferometer

Sketch



Measurements

- The heterodyne receiver measures the incoming electric field $E\cos(2\pi\nu t)$
- The correlator is a multiplier followed by a time integrator:

$$r = < E_1 \cos(2\pi\nu t)\; E_2 \cos(2\pi\nu t) > = E_1\, E_2$$

- We have measured the spatial correlation of the signal!
- ...

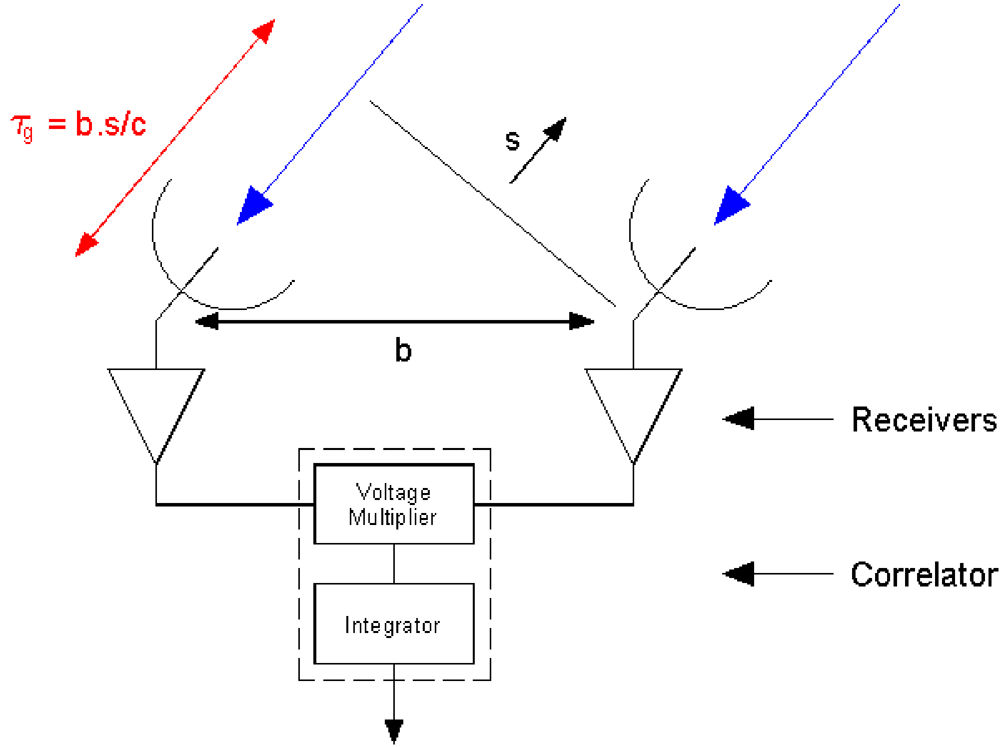
Measurements

- The heterodyne receiver measures the incoming electric field $E\cos(2\pi\nu t)$
- The correlator is a multiplier followed by a time integrator:

$$r = < E_1 \cos(2\pi\nu t) \; E_2 \cos(2\pi\nu t) > = E_1 \, E_2$$

- We have measured the spatial correlation of the signal!
- **But we have forgotten the geometrical delay**

The ideal interferometer
Sketch



The ideal interferometer
Measurements

- There is a **geometrical delay** τ_g between the two antennas $\longrightarrow$ **more complex** experiment than the Young's hole
- Correlator output:

$$
\begin{aligned}
r &= \; < E_1 \cos(2\pi\nu t) \; E_2 \cos(2\pi\nu t) > = E_1 \, E_2 \\
r &= \; < E_1 \cos(2\pi\nu(t-\tau_g)) \; E_2 \cos(2\pi\nu t) > \\
  &= \; E_1 E_2 \cos(2\pi\nu\tau_g)
\end{aligned}
$$

Measurements

- Correlator output: $r = E_1 E_2 \cos(2\pi\nu\tau_g)$
- τ_g varies slowly with time (Earth rotation) $\longrightarrow$ **fringes**
- Natural fringe rate:

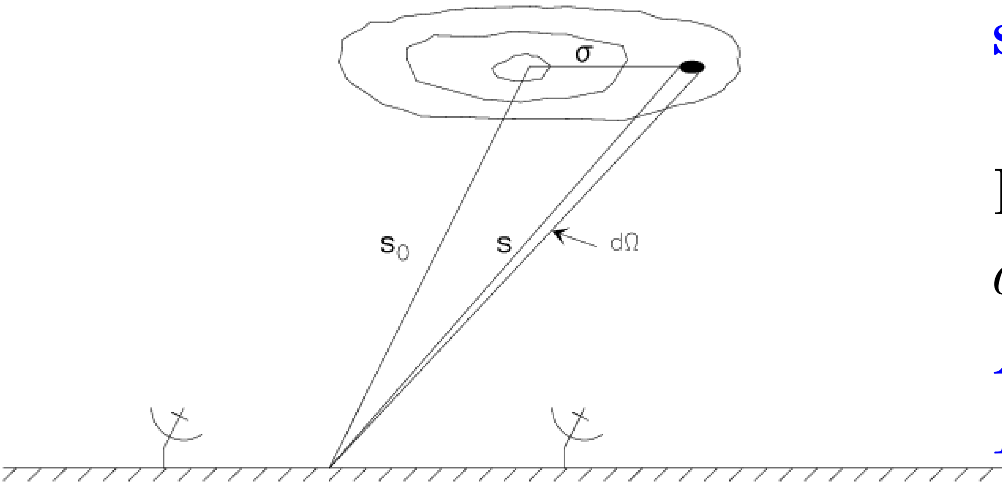
$$\tau_g = \frac{\mathbf{b.s}}{c} \qquad \nu\,\frac{d\tau_g}{dt} \simeq \Omega_{earth}\,\frac{\mathrm{b}\nu}{c}$$

$\sim$ 50 Hz for $b = 800$ m and $\nu = 250$ GHz

Measurements

- Correlator output: $r = E_1 E_2 \cos(2\pi\nu\tau_g)$
- τ_g varies slowly with time (Earth rotation) $\longrightarrow$ **fringes**
- τ_g is **known** from the antenna position, source direction, time $\longrightarrow$ could be corrected
- Problems: the source is **not a point source** ($\int \tau_g$?)
  the signal is **not monochromatic** ($\int \nu$?)

The ideal interferometer
Source size



$\mathbf{s} = \mathbf{s_o} + \sigma$

Power received from $d\Omega$: $A(\mathbf{s})I(\mathbf{s})d\Omega$

$A(s)$ = beam

$I(s)$ = source

Correlator output: $r = E_1 E_2 \cos(2\pi\nu\tau_g)$

$$r = A(\mathbf{s})I(\mathbf{s})d\Omega\cos(2\pi\nu\tau_g(\mathbf{s}))$$

The ideal interferometer
Source size

- Correlator output (integrated):

$$R = \int_{Sky} A(\mathbf{s}) I(\mathbf{s}) \cos(2\pi\nu \mathbf{b}.\mathbf{s}/c)\, d\Omega$$
$$= |V| \cos(2\pi\nu\tau_g - \varphi_{\mathrm{V}})$$

- **Complex visibility:**

$$V = |V| e^{i\varphi_{\mathrm{V}}} = \int_{Sky} A(\sigma) I(\sigma) e^{-2i\pi\nu \mathbf{b}.\sigma/c} d\Omega$$

The ideal interferometer

Source size

$$
\begin{aligned}
R &= \int_{Sky} A(\mathbf{s}) I(\mathbf{s}) \cos(2\pi\nu \mathbf{b}.\mathbf{s}/c)\, d\Omega \\
&= \cos\left(2\pi\nu\frac{\mathbf{b}.\mathbf{s}_o}{c}\right) \int_{Sky} A(\sigma) I(\sigma) \cos(2\pi\nu \mathbf{b}.\sigma/c) d\Omega \\
&- \sin\left(2\pi\nu\frac{\mathbf{b}.\mathbf{s}_o}{c}\right) \int_{Sky} A(\sigma) I(\sigma) \sin(2\pi\nu \mathbf{b}.\sigma/c) d\Omega \\
&= \cos\left(2\pi\nu\frac{\mathbf{b}.\mathbf{s}_o}{c}\right) |V| \cos\varphi_{\mathrm{V}} - \sin\left(2\pi\nu\frac{\mathbf{b}.\mathbf{s}_o}{c}\right) |V| \sin\varphi_{\mathrm{V}} \\
&= |V| \cos(2\pi\nu\tau_g - \varphi_{\mathrm{V}})
\end{aligned}
$$

The ideal interferometer
Summary

- Correlator output:

$$r = \; < E_1 \cos(2\pi\nu t) \;\; E_2 \cos(2\pi\nu t) > \; = E_1 \, E_2$$

$$r = E_1 E_2 \cos(2\pi\nu\tau_g) \qquad \longleftarrow \text{delay}$$

$$R = |V| \cos(2\pi\nu\tau_g - \varphi_{\mathrm{V}}) \qquad \longleftarrow \text{source size}$$

- Complex visibility V resembles a Fourier Transform:

$$V = |V| e^{i\varphi_{\mathrm{V}}} = \int_{Sky} A(\sigma) I(\sigma) e^{-2i\pi\nu \mathbf{b}.\sigma/c} d\Omega$$

- **3D version of van Cittert–Zernike**

Bandwidth

- Integrating over a finite bandwidth $\Delta\nu$,

$$R = \frac{1}{\Delta\nu}\int_{\nu_0-\Delta\nu/2}^{\nu_0+\Delta\nu/2} |V|\cos(2\pi\nu\tau_g - \varphi_{\rm V})\ d\nu$$
$$= |V|\cos(2\pi\nu_0\tau_g - \varphi_{\rm V})\ \frac{\sin(\pi\Delta\nu\tau_g)}{\pi\Delta\nu\tau_g}$$

- The fringe visibility is attenuated by a $\sin(x)/x$ envelope (= bandwidth pattern) which falls off rapidly.

Bandwidth

$$R = |V| \cos(2\pi\nu_0\tau_g - \varphi_{\mathrm{V}}) \, \frac{\sin(\pi\Delta\nu\tau_g)}{\pi\Delta\nu\tau_g}$$

- Tracking a source requires **compensation of the geometrical delay** = temporal coherence
- This can be achieved by introducing an **instrumental delay** in the correlator

Summary

- Correlator output:

$$r = \; < E_1 \cos(2\pi\nu t) \; E_2 \cos(2\pi\nu t) > \; = E_1 \, E_2$$

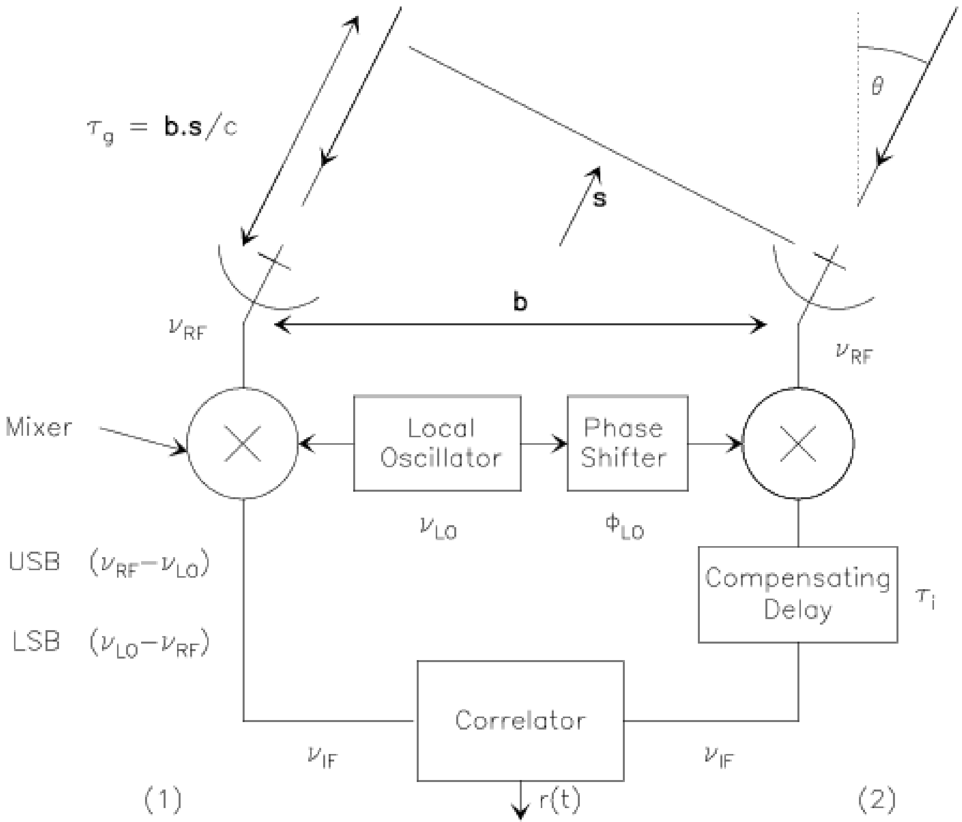
$$r = E_1 E_2 \cos(2\pi\nu\tau_g) \qquad \longleftarrow \text{delay}$$

$$R = |V| \cos(2\pi\nu\tau_g - \varphi_{\mathrm{V}}) \qquad \longleftarrow \text{source size}$$

$$R = |V| \cos(2\pi\nu_0\tau_g - \varphi_{\mathrm{V}}) \, \frac{\sin(\pi\Delta\nu\tau_g)}{\pi\Delta\nu\tau_g} \qquad \longleftarrow \text{bandwidth}$$

- We must compensate the delay to obtain $R = |V| \cos(\varphi_{\mathrm{V}})$

The real interferometer
Sketch



The real interferometer
Heterodyne detection

- In the receiver **mixer**, the incident electic field is combined with a **local oscillator** signal

$$
\begin{aligned}
U(t) &= E\cos\left(2\pi\nu t + \varphi\right) \\
U_{\mathrm{LO}}(t) &= E_{\mathrm{LO}}\cos\left(2\pi\nu_{\mathrm{LO}} t + \varphi_{\mathrm{LO}}\right) \\
\nu_{\mathrm{LO}} &\simeq \nu
\end{aligned}
$$

- The mixer is a **non-linear** element:

$$I(t) = a_0 + a_1(U + U_{\mathrm{LO}}) + a_2(U + U_{\mathrm{LO}})^2 + a_3(\ldots)^3 + \ldots$$

The real interferometer

Heterodyne detection

- There are terms at various frequencies and harmonics
- A **filter** selects the frequencies such that;

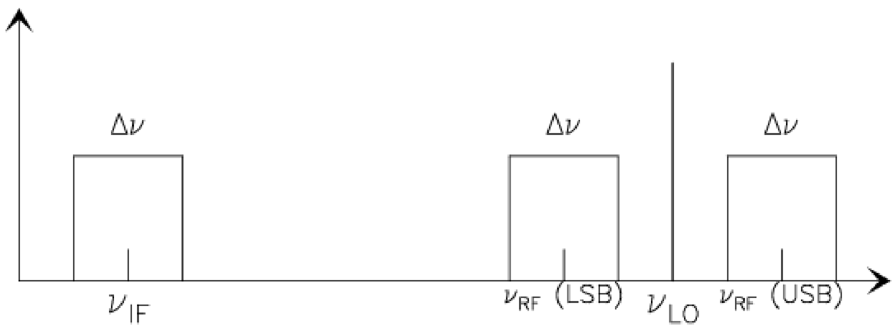
$$\nu_{\rm IF} - \Delta\nu/2 \leq |\nu - \nu_{\rm LO}| \leq \nu_{\rm IF} + \Delta\nu/2$$

- $\nu_{\rm IF}$ is the **intermediate frequency**
- $\nu_{\rm IF}$ such that amplifiers and transport elements available
- PdBI: $\nu_{\rm IF}$ = 4–8 GHz, ALMA: $\nu_{\rm IF}$ =4–12 GHz

The real interferometer
Heterodyne detection

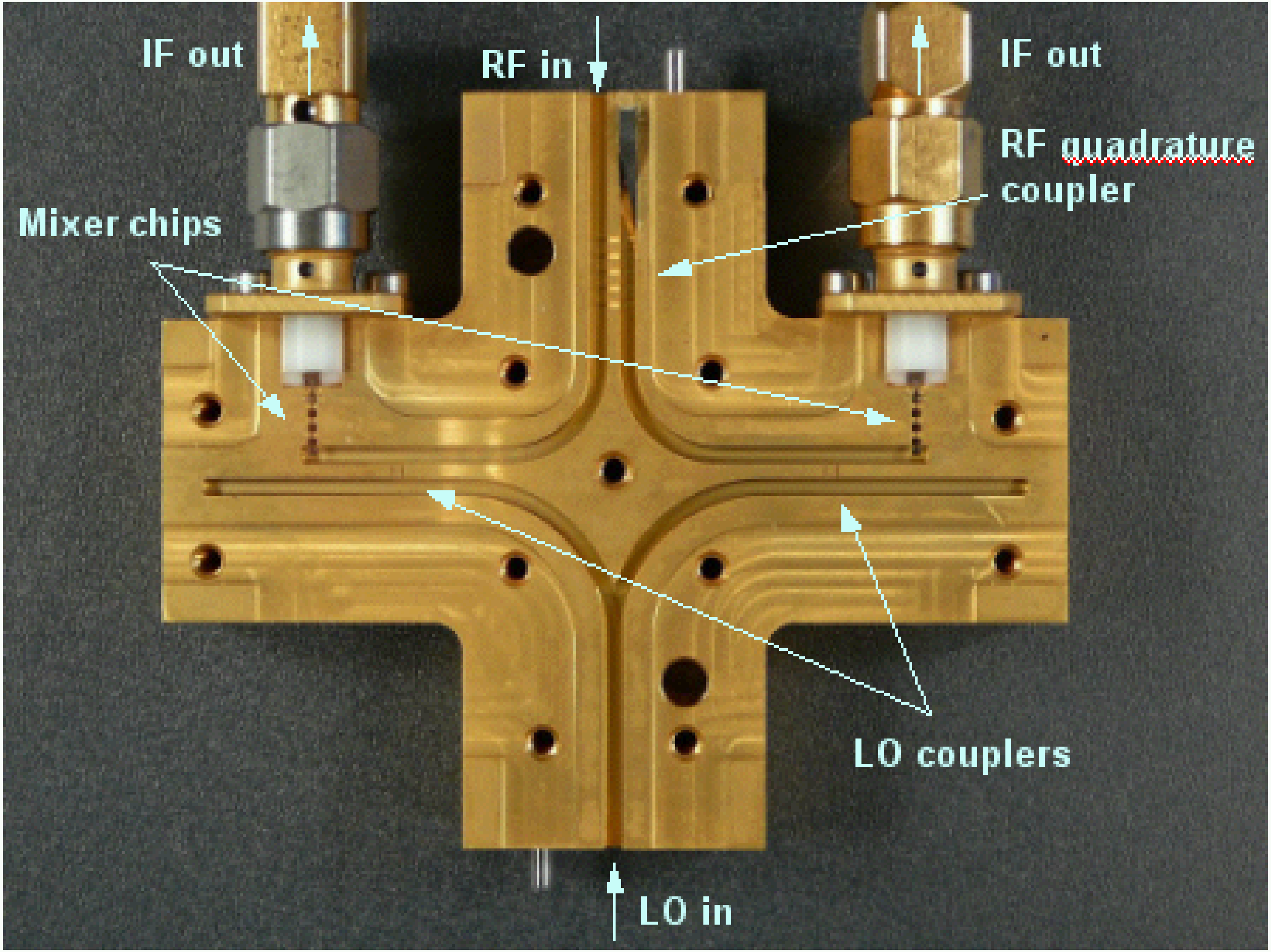
- The receiver output is

$$I(t) \propto E\, E_{\mathrm{LO}} \cos\left(\pm \left(2\pi(\nu - \nu_{\mathrm{LO}})t + \varphi - \varphi_{\mathrm{LO}}\right)\right)$$

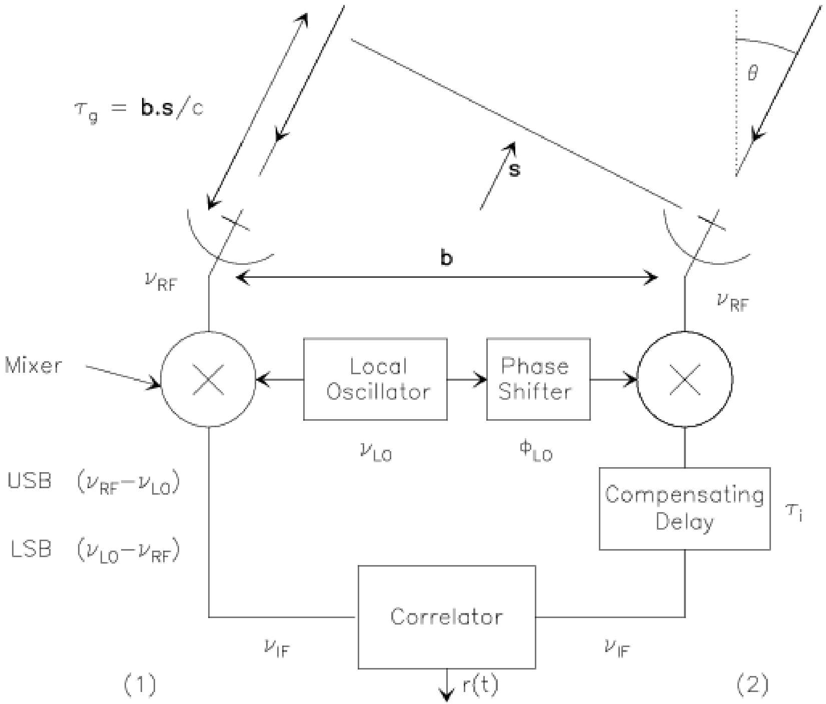


Heterodyne detection

- **DSB** receivers accept both LSB and USB frequencies, i.e. their output is the sum of LSB and USB
- **SSB** receivers accept only LSB or USB (response very strongly frequency dependant)
- **2SB** receivers are 2 DSB receivers combined such that the two bands are independently output (and processed)


IF out
RF in
IF out
RF quadrature
coupler
Mixer chips
LO couplers
LO in

Delay tracking



- A compensating delay is introduced in one of the branch of the interferometer, **on the IF signal**
- Equivalent to the delay lines in IR interferometers

The real interferometer

Delay tracking

- Phases of the two signals (USB):

$$\varphi_1 = 2\pi\nu\tau_g \qquad \varphi_1 = 2\pi\nu\tau_g = 2\pi(\nu_{\mathrm{LO}} + \nu_{\mathrm{IF}})\tau_g$$

$$\varphi_2 = 0 \qquad \varphi_2 = 2\pi\nu_{\mathrm{IF}}\tau_i$$

- Correlator output:

$$R = |V|\cos(2\pi\nu\tau_g - \varphi_{\mathrm{V}})$$

$$R = |V|\cos(\varphi_1 - \varphi_2 - \varphi_{\mathrm{V}})$$

$$R = |V|\cos(2\pi\nu_{\mathrm{LO}}\tau_g - \varphi_{\mathrm{V}})$$

The real interferometer

Fringe Stopping

- Delay tracking not enough because applied on the IF
- Solution: in addition to delay tracking, **rotate the phase of the local oscillator** such that at any time:

$$\varphi_{\mathrm{LO}}(t) = 2\pi\nu_{\mathrm{LO}}\tau_g(t)$$

- τ_g is computed for a reference position = **phase center**
- Phase center = pointing center in practice, though not mandatory

Fringe stopping

- Phases of the two signals (USB):

$$\begin{aligned}\varphi_1 &= 2\pi\nu\tau_g = 2\pi(\nu_{\mathrm{LO}} + \nu_{\mathrm{IF}})\tau_g \\ \varphi_2 &= 2\pi\nu_{\mathrm{IF}}\tau_i + \varphi_{\mathrm{LO}} \\ \varphi_{\mathrm{LO}} &= 2\pi\nu_{\mathrm{LO}}\tau_g\end{aligned}$$

- Correlator output:

$$\begin{aligned}R &= |V|\cos(\varphi_1 - \varphi_2 - \varphi_{\mathrm{V}}) \\ R &= |V|\cos(\varphi_{\mathrm{V}})\end{aligned}$$

The real interferometer

Complex correlator

- After fringe stopping:

$$r_r = |V| \cos(-\varphi_{\rm V})$$

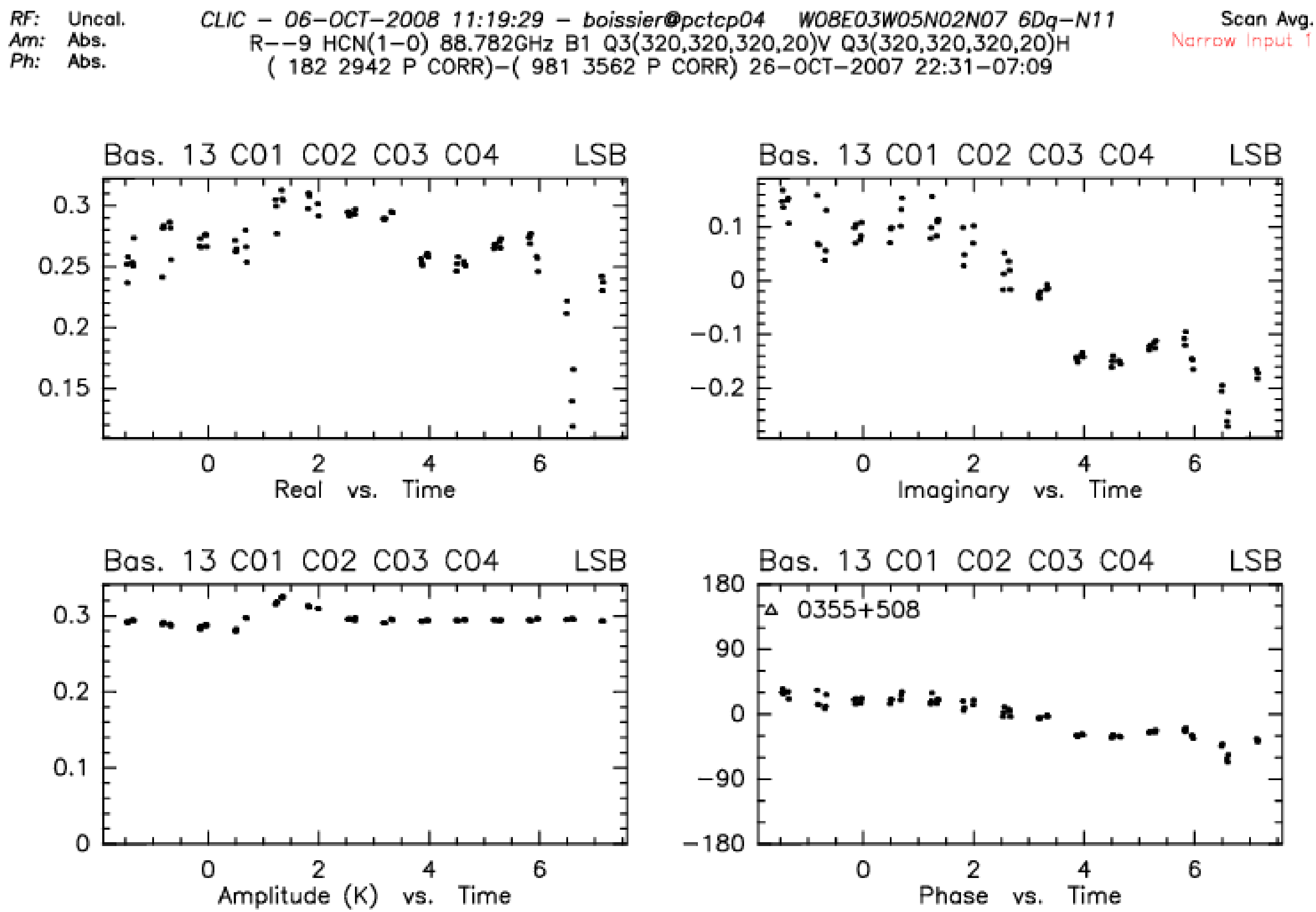
- No time/delay dependance any more $\longrightarrow$ cannot measure $|V|$ and $\varphi_{\rm V}$ separately.
- A second correlator is necessary, with one signal phase shifted by $\pi/2$:

$$r_i = |V| \sin(-\varphi_{\rm V})$$

- **The complex correlator measures directly the visibility**

Complex correlator

- The correlator measures the real and imaginary parts of the visibility. **Amplitude and phases are computed off-line.**
- Amplitude and phases have more physical sense
  - Visibility amplitude = **correlated flux**
  - The atmosphere adds a **phase** to the incoming signals $\longrightarrow$ measured phase = visibility + $\varphi_1 - \varphi_2$


RF: Uncal.
Am: Abs.
Ph: Abs.
CLIC – 06-OCT-2008 11:19:29 – boissier@pctcp04 W08E03W05N02N07 6Dq-N11
R--9 HCN(1-0) 88.782GHz B1 Q3(320,320,320,20)V Q3(320,320,320,20)H
(182 2942 P CORR)-(981 3562 P CORR) 26-OCT-2007 22:31-07:09
Scan Avg.
Narrow Input 1
Bas. 13 C01 C02 C03 C04 LSB
Real vs. Time
Bas. 13 C01 C02 C03 C04 LSB
Imaginary vs. Time
Bas. 13 C01 C02 C03 C04 LSB
Amplitude (K) vs. Time
Bas. 13 C01 C02 C03 C04 LSB
0355+508
Phase vs. Time

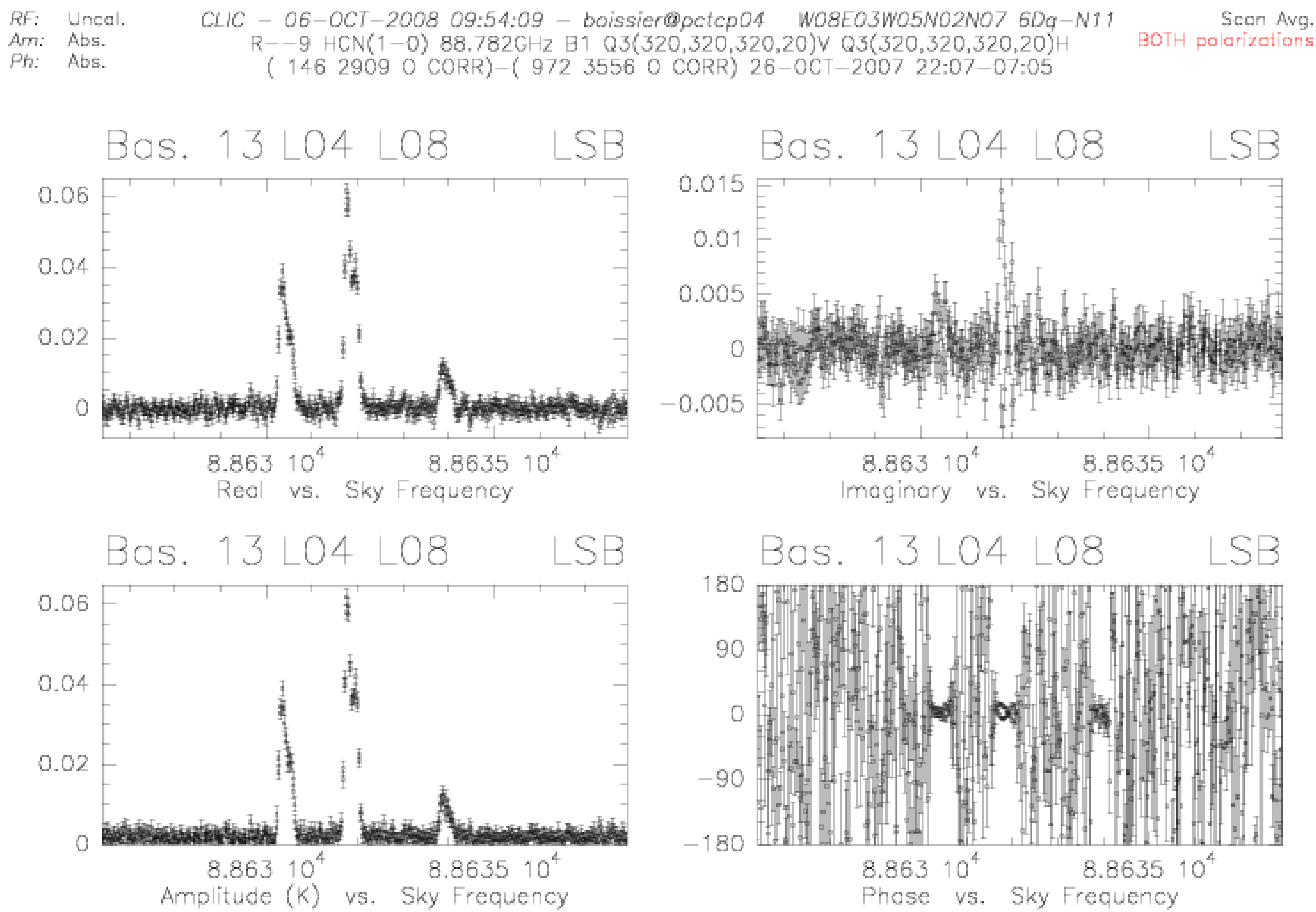
The real interferometer

Spectroscopy

- Remember the Wiener-Kichnine theorem?
- Calculate the correlation function for several delay $\delta\tau \longrightarrow$ measurement of the **temporal correlation** $\longrightarrow$ FT to get the spectra:

$$V_\nu(u, v, \nu) = \int V(u, v, \tau) e^{-2i\pi\tau\nu} d\nu$$

- Nothing to do with geometrical delay compensation – $\delta\tau \sim 1/\delta\nu$ here
- Mixed up implementation in correlator software


RF: Uncal.
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CLIC - 06-OCT-2008 09:54:09 - boissier@pctcp04 W08E03W05N02N07 6Dq-N11
R--9 HCN(1-0) 88.782GHz B1 Q3(320,320,320,20)V Q3(320,320,320,20)H
(146 2909 0 CORR)-(972 3556 0 CORR) 26-OCT-2007 22:07-07:05
Scan Avg.
BOTH polarizations
Bas. 13 L04 L08 LSB
Real vs. Sky Frequency
Bas. 13 L04 L08 LSB
Imaginary vs. Sky Frequency
Bas. 13 L04 L08 LSB
Amplitude (K) vs. Sky Frequency
Bas. 13 L04 L08 LSB
Phase vs. Sky Frequency

Goal

Interferometry

<u>Implementing the van Cittert–Zernike theorem</u>

1. Build a device that measures the spatial autocorrelation of the incoming signal $\longrightarrow$ **2-elements interferometer**
2. <u>Do it for all possible scales</u> $\longrightarrow$ **N antennas**
3. Take the FT and get an image of the brightness distribution $\longrightarrow$ **software**

Complex visibility

- Complex visibility:

$$V = |V|e^{i\varphi_{\mathrm{V}}} = \int_{Sky} A(\sigma) I(\sigma) e^{-2i\pi\nu \mathbf{b}.\sigma/c} d\Omega$$

- Going from 3-D to 2-D? ...some algrebra...
- OK providing that:

$$(\text{max. field of view})^2 \times \text{max. baseline} \ll 1$$

$$\Longrightarrow \frac{(\text{max. field of view})^2}{\text{resolution}} \ll 1$$

Aperture synthesis

Complex visibility

$$V(u,v) = \int_{Sky} A(\ell,m) I(\ell,m) e^{-2i\pi\nu(u\ell+vm)} d\Omega$$

- uv plane is perpendicular to the source direction, **fixed wrt source** $\longrightarrow$ **back to Young's hole**
- Price: limit on the field of view
- Approximation **ok in (sub)mm domain**, problem at wavelengths $>$ cm

Aperture synthesis
(Field of view)

- Field of view is limited by
  - the **antenna primary beam**: the interferometer measures $A \times I$
  - the 2D visibility approximation
  - the frequency averaging (bandwidth)
  - the time averaging (integration)

    $\hookrightarrow$ averaging in the uv plane; possible only if limited field of view

Aperture synthesis
(Field of view)

- Values for Plateau de Bure

θ_s	ν (GHz)	2-D Field	0.5 GHz Bandwidth	1 Min Averaging	Primary Beam
$5''$	80	$5'$	$80''$	$2'$	$60''$
$2''$	80	$3.5'$	$30''$	$45''$	$60''$
$2''$	230	$3.5'$	$1.5'$	$45''$	$24''$
$0.5''$	230	$1.7'$	$22''$	$12''$	$24''$

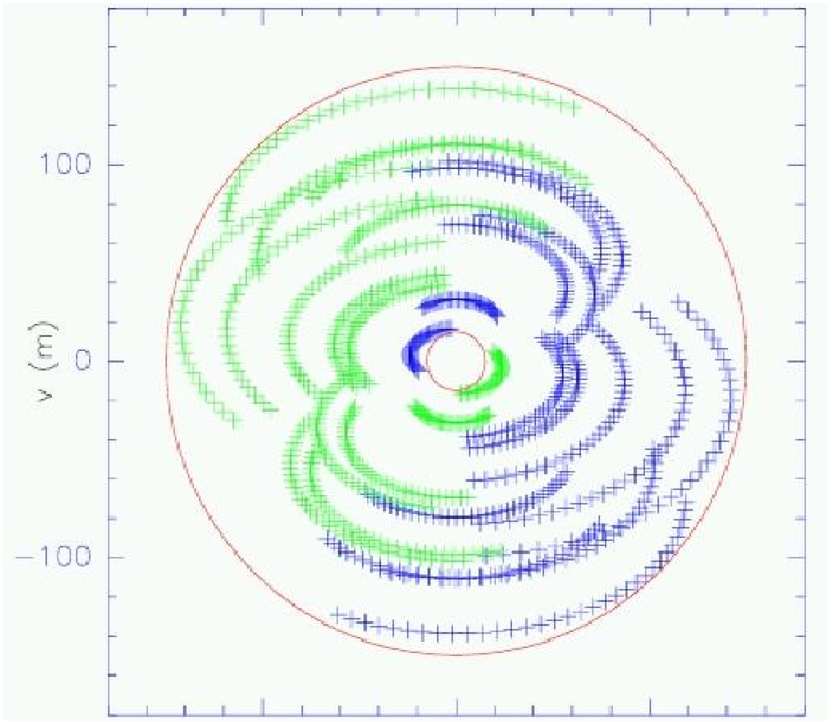
- Problem with 2D field: software; with bandwith: split the data for imaging; with time averaging: dump faster.
- **Primary beam is the main limit on the FOV**

Aperture synthesis
uv plane

- uv plane is perpendicular to the source direction, **fixed wrt source** $\longrightarrow$ **back to Young's hole**
- (u, v) is the 2–antennas **vector** baseline projected on the plane perpendicular to the source
- (u, v) are **spatial frequencies**
- ... Earth rotation ... (spherical trigonometry) ...
- (u, v) describe an **ellipse** in the uv plane (for $\delta = 0\,\mathrm{deg}$, a line)

Aperture synthesis

uv plane coverage



Max. baseline gives the angular resolution

Aperture synthesis
Transfer function

Single-dish observations

$$\begin{array}{ccc} \text{Aperture function} & \rightleftharpoons & \text{Voltage pattern} \\ & & \Big\downarrow \, |\cdot|^2 \\ & \rightleftharpoons & \text{Power pattern } B(\ell, m) \\ & & \textcolor{blue}{\mathbf{= Primary\ beam}} \end{array}$$

Transfer function

Single-dish observations

$$\begin{array}{ccc} \text{Aperture function} & \rightleftharpoons & \text{Voltage pattern} \\ \color{red}{\star}\big\downarrow & & \big\downarrow |\cdot|^2 \\ \color{red}{\text{Transfer function } T(u,v)} & \rightleftharpoons & \text{Power pattern } B(\ell, m) \\ & & \color{blue}{\mathbf{= Primary\ beam}} \end{array}$$

Transfer function describes how spatial frequencies are transmitted by the telescope

Aperture synthesis
Transfer function

<u>Interferometers</u>

$$\begin{tikzcd}
\text{Aperture function} \arrow[r, phantom, "\rightleftharpoons"] \arrow[d, "\star"'] & \text{Voltage pattern} \arrow[d, "|\cdot|^2"] \\
\text{Transfer function } \mathbf{T}(\mathbf{u},\mathbf{v}) \arrow[r, phantom, "\rightleftharpoons"] & \text{Power pattern } B(\ell, m)
\end{tikzcd}$$

$= $ **Primary beam**

Aperture synthesis = sample directly the transfer function

Sensitivity
Radiometric formula

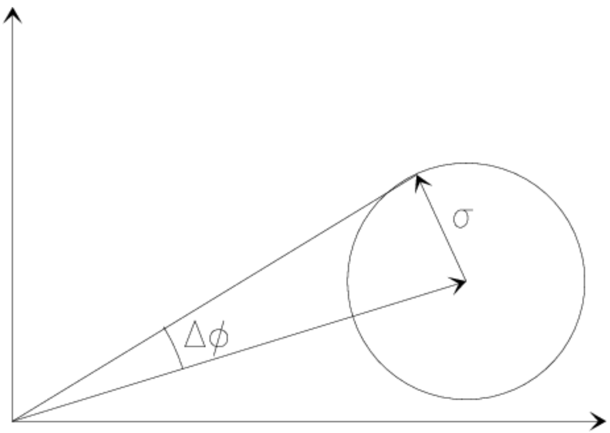
- Measurement of visibilities is limited by noise emitted by atmosphere, antenna, ground, receivers.
- The rms noise for the baseline ij is given by:

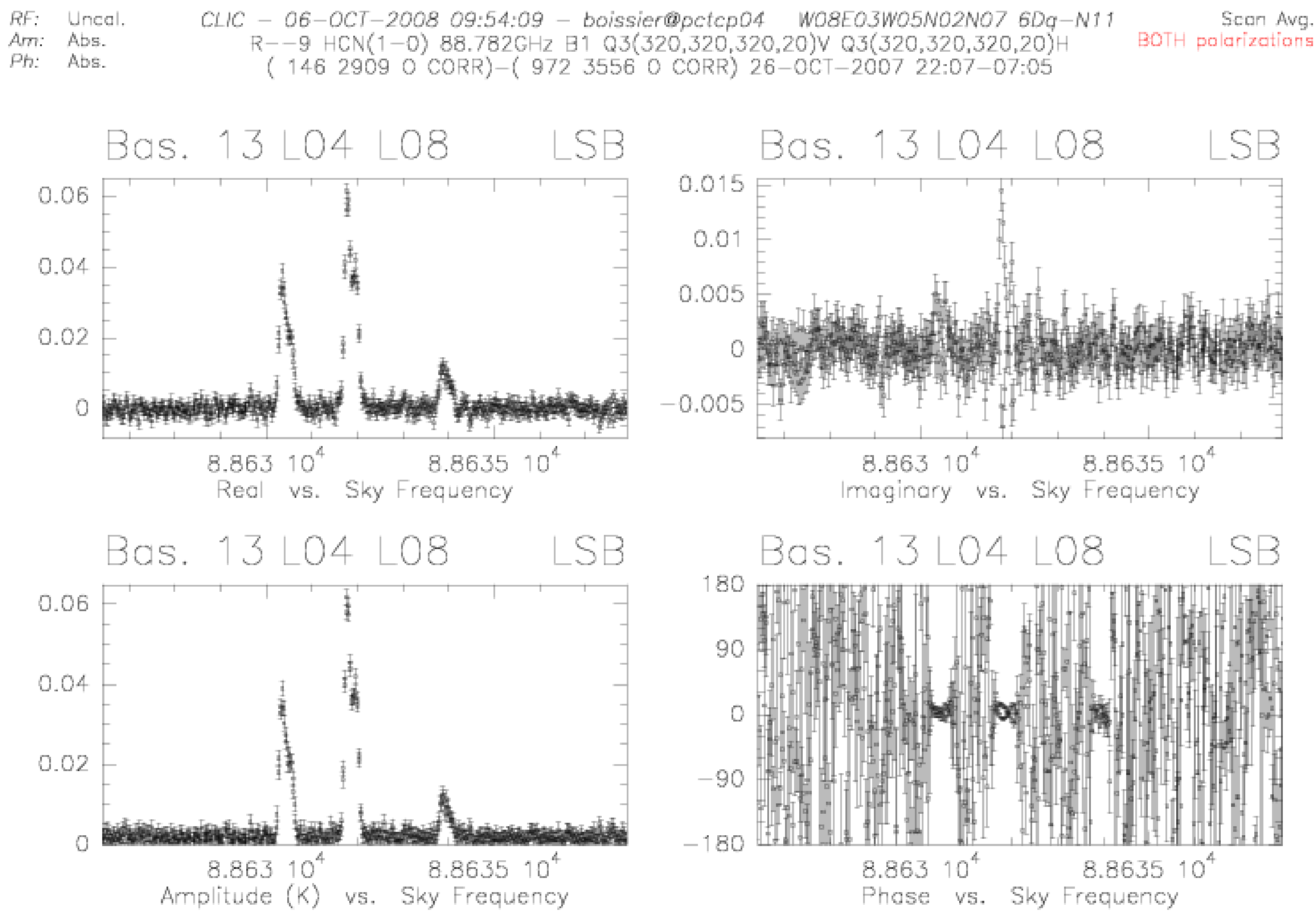
$$\delta S_{ij} = \frac{2k}{A\eta_{\mathrm{A}}\eta_{\mathrm{Q}}\eta_{\mathrm{P}}} \cdot \frac{\sqrt{T_{\mathrm{SYS}i}T_{\mathrm{SYS}j}}}{\sqrt{2B\,T}}$$

  - A antenna physical aperture
  - η_{A} antenna aperture efficiency
  - η_{Q} efficiency for the correlator
  - $T_{\mathrm{SYS}i}$ system noise temperature (single dish)
  - B bandwidth
  - T integration time
  - η_{P} phase decorrelation factor (LO jitter)

Radiometric formula

- This is the noise on the **real** and on the **imaginary** parts of the visibilities (measured independently)
- This is also the noise on the **amplitude** S
- Noise on the phase more complex, of the order of σ/S

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Amplitude (K) vs. Sky Frequency
Bas. 13 L04 L08 LSB
Phase vs. Sky Frequency

Sensitivity
Radiometric formula

- For N identical antenna/receivers, i.e. $N(N-1)/2$ baselines, the **point-source** sensitivity is:

$$\delta S = \frac{2k}{A \eta_{\mathrm{A}} \eta_{\mathrm{Q}} \eta_{\mathrm{P}}} \cdot \frac{T_{\mathrm{SYS}}}{\sqrt{N(N-1)\, B\, T}}$$

- Scales as $\sim 1/N$
- Sensitivity to extended sources depends on angular resolution

Sensitivity
Phase decorrelation

- **Short term phase errors** in the local oscillators (jitter) will cause a **decorrelation** of the signal and reduce the visibility amplitude by a factor

$$\eta_{\mathrm{P}}(12) = e^{-({\sigma_1}^2+{\sigma_2}^2)/2} = \eta_1\eta_2$$

- Requirements:

η_1	0.99	0.98	0.95	0.90
σ_1 (degrees)	8.1	11.5	18.3	26.4

Sensitivity

Phase decorrelation

- $\eta_{\mathrm{P}} = 0.9 \longrightarrow \eta_1 = 0.95 \longrightarrow \sigma_1 = 18\,\mathrm{deg}$
- PdBI: LO derived from a reference at 1.8 GHz
- Phase stability required $= \sigma_1$ (1.8 GHz/230 GHz) $\sim 0.15^\circ$
- **Very stable** oscillators are required
- Phase decorrelation due to the atmosphere more severe problem

Summary

Other instrumental issues

- Antenna position measurements, to get the delay, u, v
- Phase lock systems to control φ_{LO}
- Real-time monitoring and correction of the phase offset in the cables or fibers
- Antenna deformations, e.g. thermal expansion (delay)
- Accurate focus measurements (delay)
- Atmospheric phase monitoring
- ...

Summary

It works!