mm interferometers

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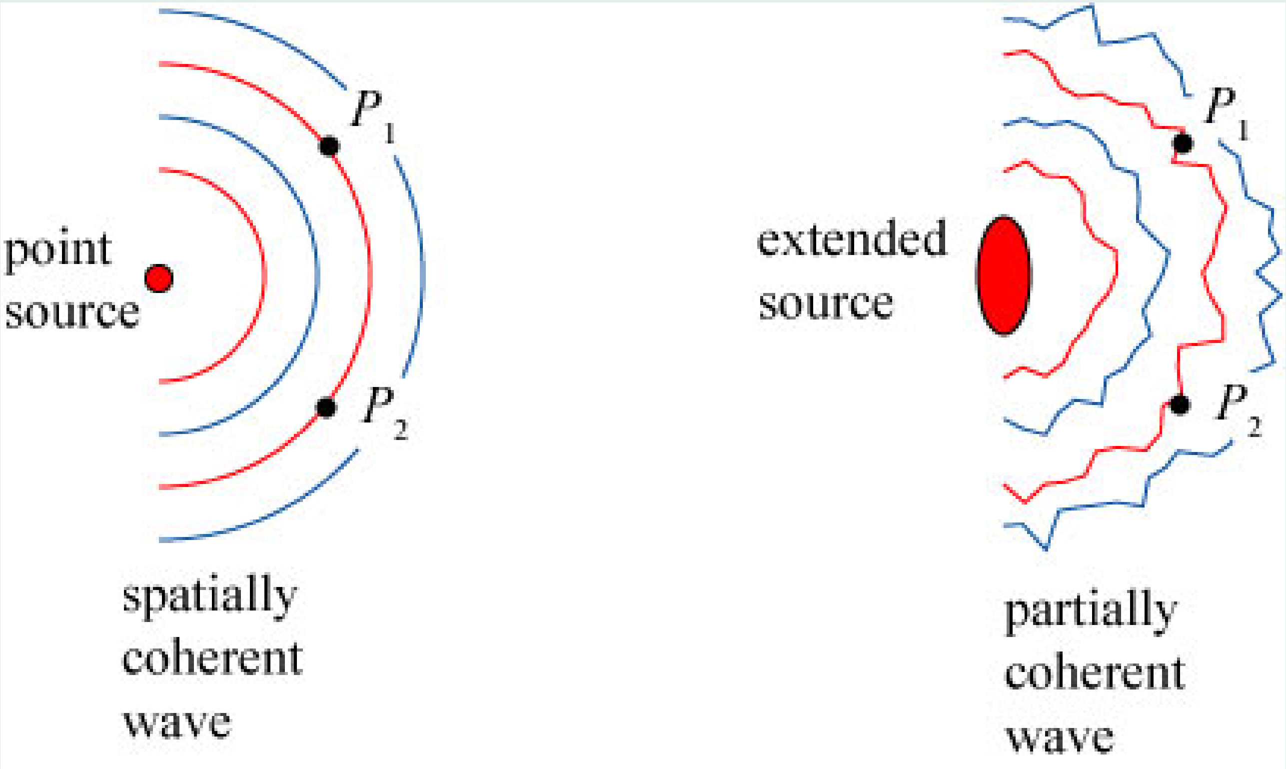
Outline

- The van Cittert–Zernike theorem
- The ideal interferometer
  - $\hookrightarrow$ geometrical delay, source size, bandwidth
- The real interferometer
  - $\hookrightarrow$ heterodyne receivers, delay correction, correlators
- Aperture synthesis
  - $\hookrightarrow$ uv plane, field of view, transfer function
- Sensitivity

van Cittert–Zernike theorem



van Cittert–Zernike theorem

- Source at infinite distance; no spatial coherence; homogeneous medium between source and measure; measure in plane perp. to the line of sight
- **Spatial autocorrelation of measured field = FT(source brightness)**

$$< E(x_1)\, E(x_2) >= \Sigma(u) \rightleftharpoons S(\alpha)$$

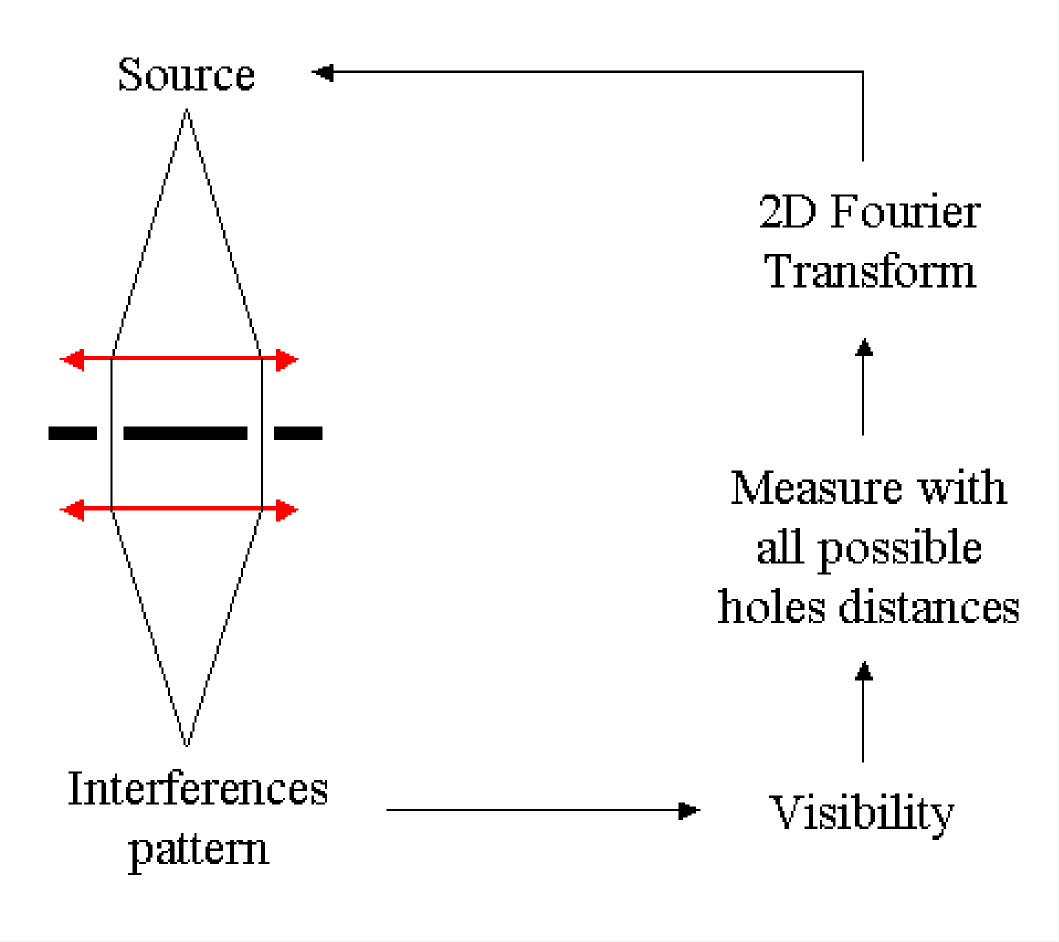
$< E(x_1)\, E(x_2) >=$ spatial correlation of incoming field

$u = x_1 - x_2 =$ spatial frequency

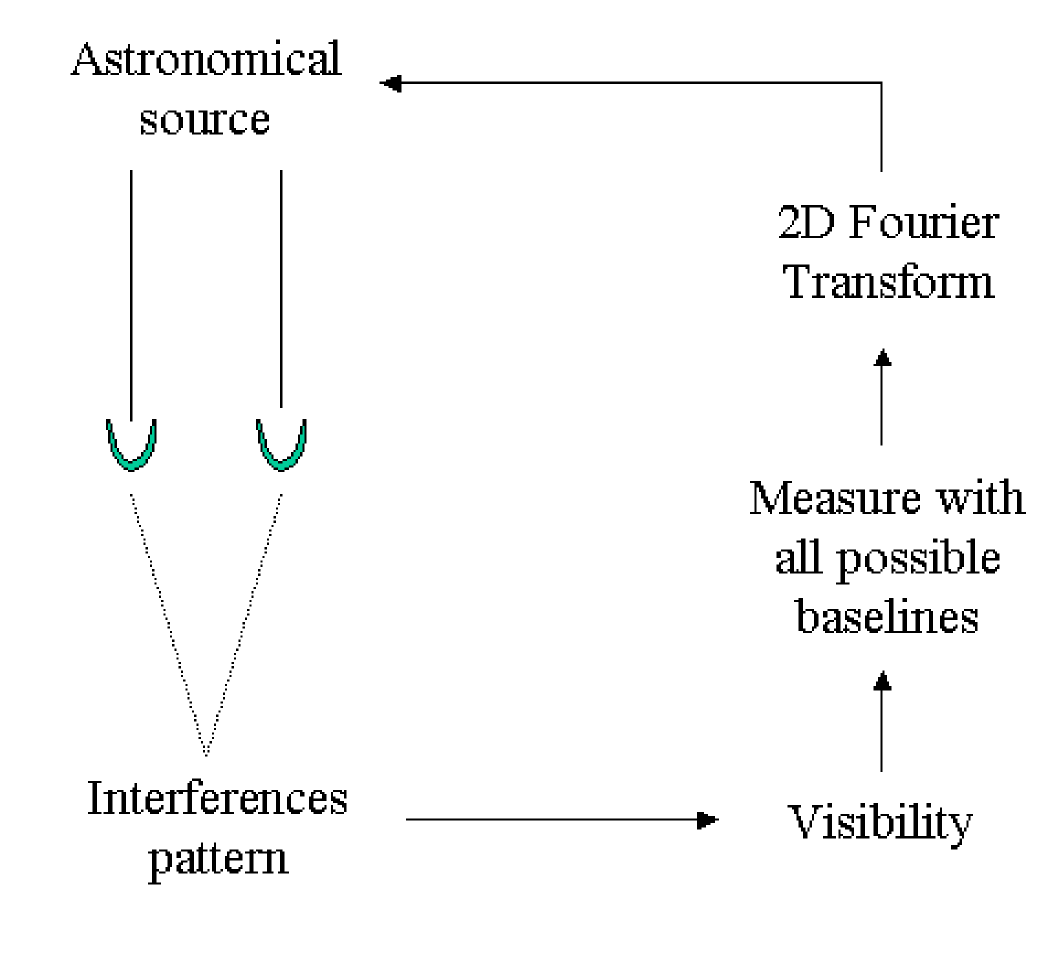
$\alpha =$ angular direction

$S(\alpha) =$ brightness distribution

Young's holes



Astronomical source



Implementing the van Cittert–Zernike theorem

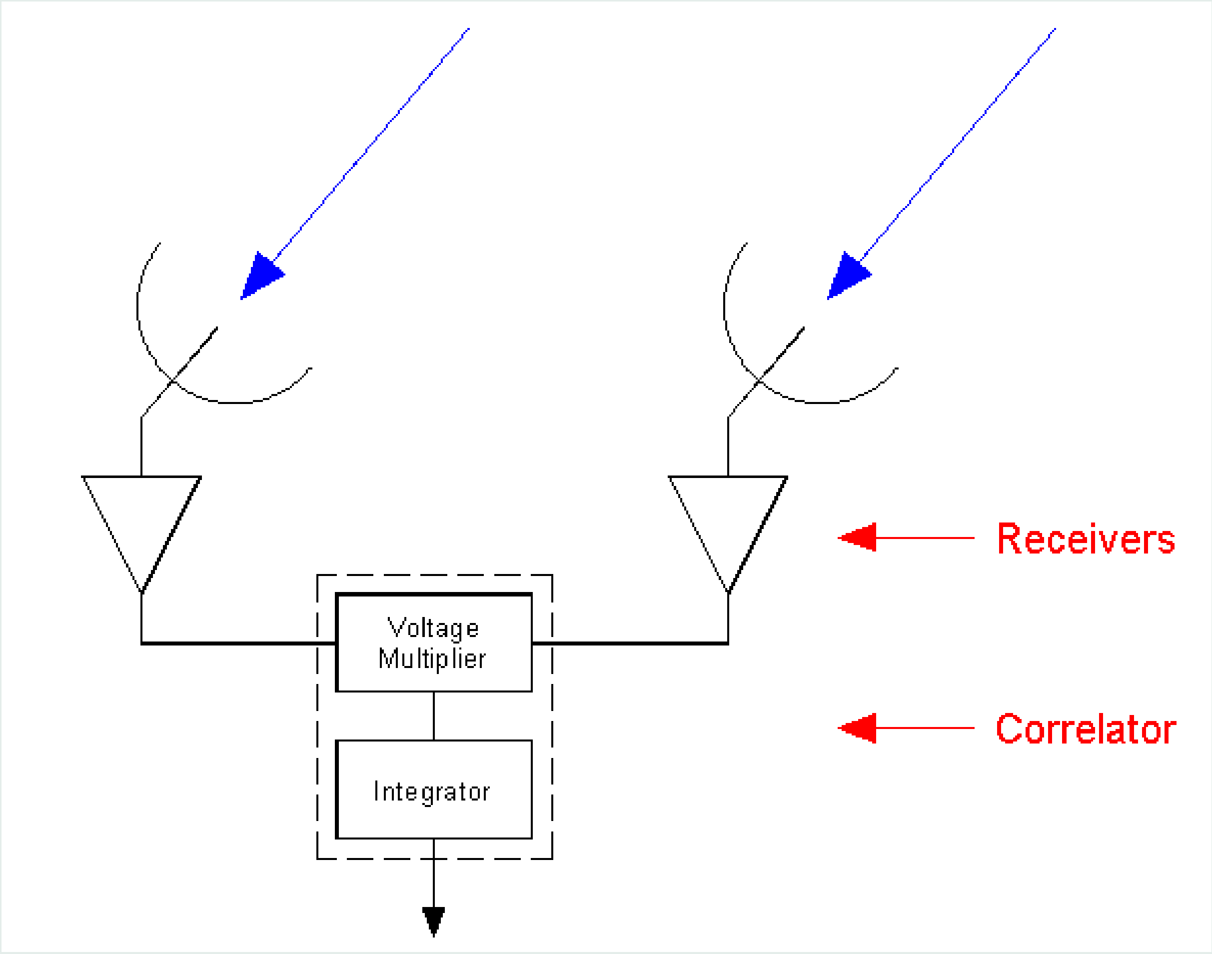
1. Build a device that measures the spatial autocorrelation of the incoming signal
2. Do it for all possible baselines
3. Take the FT and get an image of the brightness distribution

van Cittert–Zernike theorem

Implementing the van Cittert–Zernike theorem

1. Build a device that measures the spatial autocorrelation of the incoming signal $\longrightarrow$ **2-elements interferometer**
2. Do it for all possible scales $\longrightarrow$ **N antennas**
3. Take the FT and get an image of the brightness distribution $\longrightarrow$ **software**

Sketch



Measurements

- The heterodyne receiver measures the incoming electric field $E\cos(2\pi\nu t)$
- The correlator is a multiplier followed by a time integrator:

$$r = < E_1 \cos(2\pi\nu t)\; E_2 \cos(2\pi\nu t) > = E_1\, E_2$$

- We have measured the spatial correlation of the signal!
- ...

Measurements

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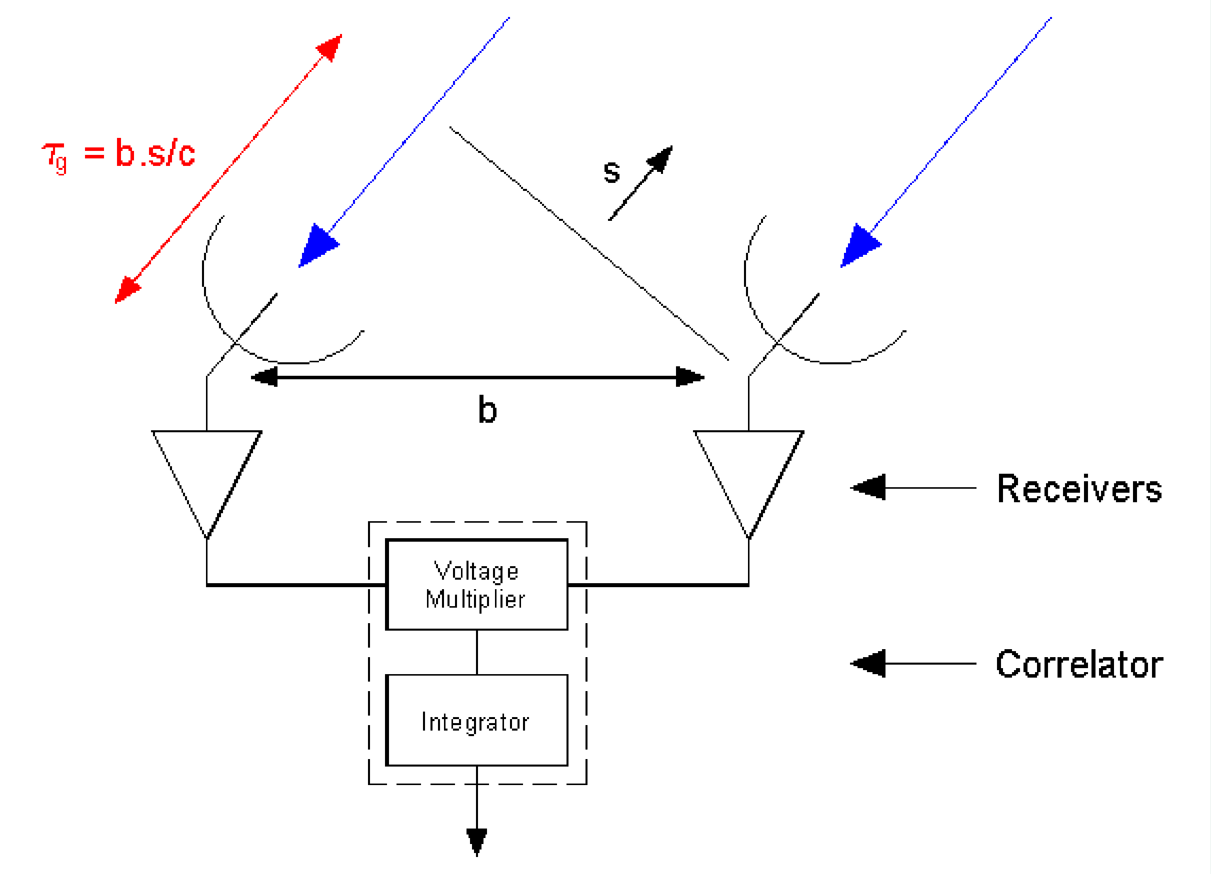
$$r = < E_1 \cos(2\pi\nu t) \; E_2 \cos(2\pi\nu t) > = E_1 \, E_2$$

- We have measured the spatial correlation of the signal!
- **But we have forgotten the geometrical delay**

The ideal interferometer
Sketch



Measurements

- There is a **geometrical delay** τ_g between the two antennas
  $\longrightarrow$ **more complex** experiment than the Young's holes
- Correlator output:

$$r = \; < E_1 \cos(2\pi\nu t) \; E_2 \cos(2\pi\nu t) > \; = E_1 \, E_2$$

$$\begin{aligned} r &= \; < E_1 \cos(2\pi\nu(t - \tau_g)) \; E_2 \cos(2\pi\nu t) > \\ &= E_1 E_2 \cos(2\pi\nu\tau_g) \end{aligned}$$

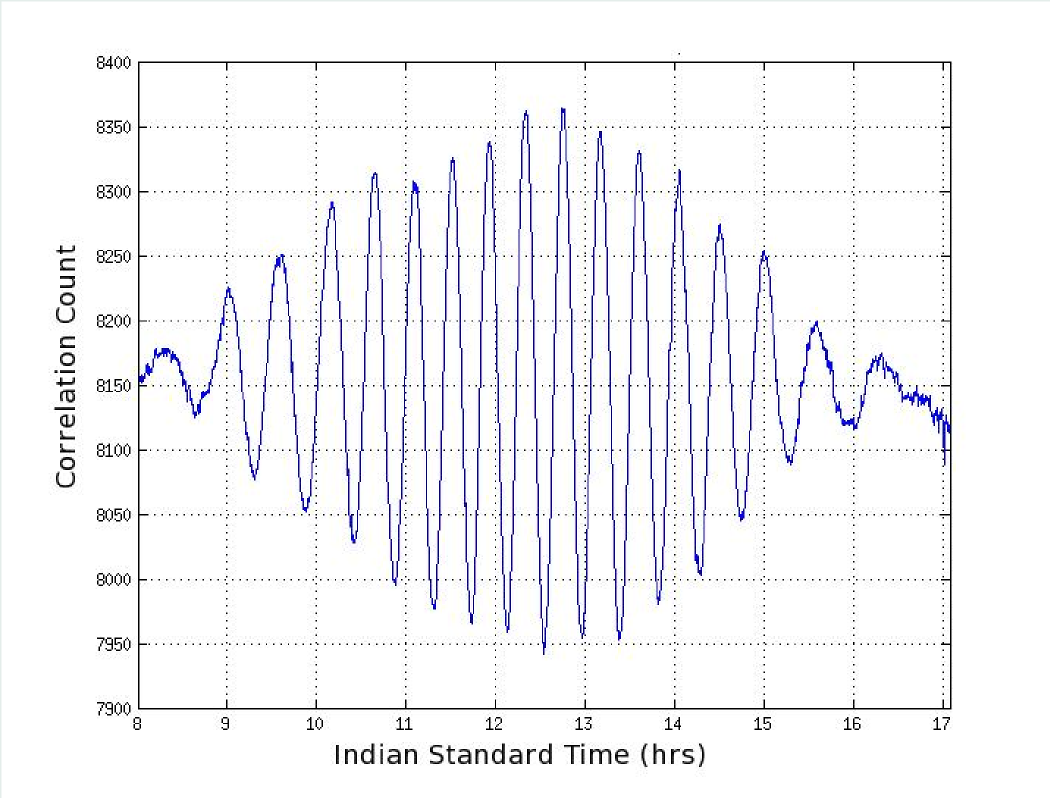
Measurements

- Correlator output: $r = E_1 E_2 \cos(2\pi\nu\tau_g)$
- τ_g varies slowly with time (Earth rotation) $\longrightarrow$ **fringes**
- Natural fringe rate:

$$\tau_g = \frac{\mathbf{b.s}}{c} \qquad\qquad \nu\,\frac{d\tau_g}{dt} \simeq \Omega_{earth}\,\frac{\mathrm{b}\nu}{c}$$

$\sim$ 50 Hz for $b = 800$ m and $\nu = 250$ GHz

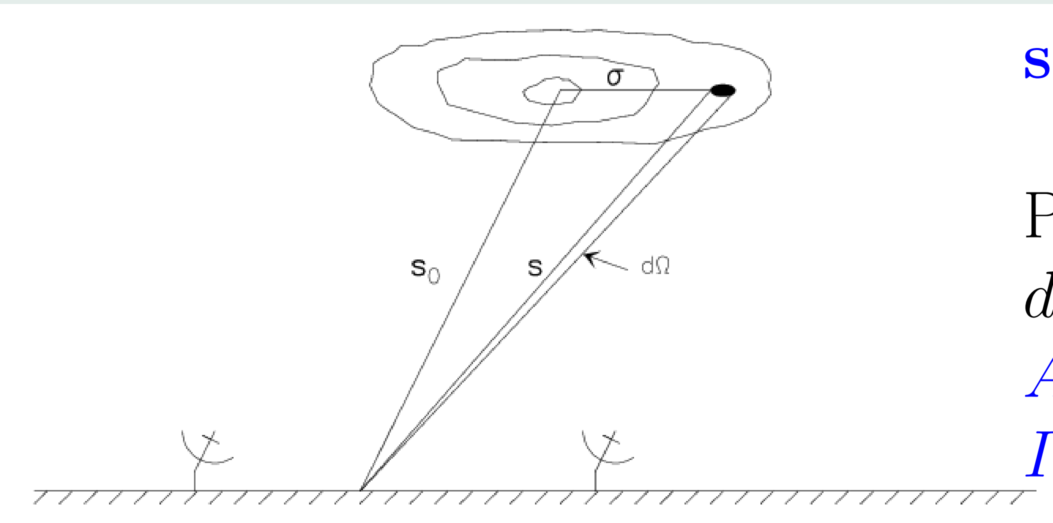
Measurements



Measurements

- Correlator output: $r = E_1 E_2 \cos(2\pi\nu\tau_g)$
- τ_g varies slowly with time (Earth rotation) $\longrightarrow$ **fringes**
- τ_g is **known** from the antenna position, source direction, time $\longrightarrow$ could be corrected
- But... the source is **not a point source**
  and the signal is **not monochromatic**

The ideal interferometer
Source size



$\mathbf{s} = \mathbf{s_o} + \sigma$

Power received from $d\Omega$: $A(\mathbf{s}) I(\mathbf{s}) d\Omega$

$A(s)$ = beam

$I(s)$ = source

Correlator output: $r = E_1 E_2 \cos(2\pi\nu\tau_g)$

$r = A(\mathbf{s}) I(\mathbf{s}) d\Omega \cos(2\pi\nu\tau_g(\mathbf{s}))$

Source size

- Correlator output integrated over source:

$$R = \int_{Sky} A(\mathbf{s}) I(\mathbf{s}) \cos(2\pi\nu\tau_g(\mathbf{s}))\, d\Omega$$
$$= |V| \cos(2\pi\nu\tau_g - \varphi_{\mathrm{V}})$$

- **Complex visibility:**

$$V = |V| e^{i\varphi_{\mathrm{V}}} = \int_{Sky} A(\sigma) I(\sigma) e^{-2i\pi\nu \mathbf{b}.\sigma/c} d\Omega$$

Source size

$$
\begin{aligned}
R &= \int_{Sky} A(\mathbf{s}) I(\mathbf{s}) \cos(2\pi\nu \mathbf{b}.\mathbf{s}/c)\, d\Omega \\
&= \cos\left(2\pi\nu\frac{\mathbf{b}.\mathbf{s}_o}{c}\right) \int_{Sky} A(\sigma) I(\sigma) \cos(2\pi\nu \mathbf{b}.\sigma/c) d\Omega \\
&- \sin\left(2\pi\nu\frac{\mathbf{b}.\mathbf{s}_o}{c}\right) \int_{Sky} A(\sigma) I(\sigma) \sin(2\pi\nu \mathbf{b}.\sigma/c) d\Omega \\
&= \cos\left(2\pi\nu\frac{\mathbf{b}.\mathbf{s}_o}{c}\right) |V| \cos\varphi_{\mathrm{V}} - \sin\left(2\pi\nu\frac{\mathbf{b}.\mathbf{s}_o}{c}\right) |V| \sin\varphi_{\mathrm{V}} \\
&= |V| \cos(2\pi\nu\tau_g - \varphi_{\mathrm{V}})
\end{aligned}
$$

The ideal interferometer
Summary

- Correlator output:

$$r = \; < E_1 \cos(2\pi\nu t) \; E_2 \cos(2\pi\nu t) > \; = E_1 \, E_2$$

$$r = E_1 E_2 \cos(2\pi\nu\tau_g) \qquad \longleftarrow \text{delay}$$

$$R = |V| \cos(2\pi\nu\tau_g - \varphi_{\rm V}) \qquad \longleftarrow \text{source size}$$

- Complex visibility V resembles a Fourier Transform:

$$V = |V| e^{i\varphi_{\rm V}} = \int_{Sky} A(\sigma) I(\sigma) e^{-2i\pi\nu\mathbf{b}.\sigma/c} d\Omega$$

The ideal interferometer

Summary

- Correlator output:

$$r = \; < E_1 \cos(2\pi\nu t) \;\; E_2 \cos(2\pi\nu t) > \; = E_1 \, E_2$$

$$r = E_1 E_2 \cos(2\pi\nu\tau_g) \qquad \longleftarrow \text{delay}$$

$$R = |V| \cos(2\pi\nu\tau_g - \varphi_{\mathrm{V}}) \qquad \longleftarrow \text{source size}$$

- **3D version of van Cittert–Zernike**
  - We do **not** measure $r = FT(I)$
  - We measure $R =$ something related to V, which resembles the FT(I)

Bandwidth

- Next problem: bandwidth
- Integrating over a finite bandwidth $\Delta\nu$

$$R = \frac{1}{\Delta\nu} \int_{\nu_0-\Delta\nu/2}^{\nu_0+\Delta\nu/2} |V| \cos(2\pi\nu\tau_g - \varphi_{\mathrm{V}})\ d\nu$$

$$= |V| \cos(2\pi\nu_0\tau_g - \varphi_{\mathrm{V}})\ \frac{\sin(\pi\Delta\nu\tau_g)}{\pi\Delta\nu\tau_g}$$

- The fringe visibility is attenuated by a $\sin(x)/x$ envelope (= bandwidth pattern) which falls off rapidly

The ideal interferometer

Summary

- Correlator output:

$$r = \ < E_1 \cos(2\pi\nu t) \ E_2 \cos(2\pi\nu t) > = E_1 \, E_2$$

$$r = \ E_1 E_2 \cos(2\pi\nu\tau_g) \qquad \longleftarrow \text{delay}$$

$$R = |V| \cos(2\pi\nu\tau_g - \varphi_{\rm V}) \qquad \longleftarrow \text{source size}$$

$$R = |V| \cos(2\pi\nu_0\tau_g - \varphi_{\rm V}) \, \frac{\sin(\pi\Delta\nu\tau_g)}{\pi\Delta\nu\tau_g} \qquad \longleftarrow \text{bandwidth}$$

- We measure R, which is related to V, which resembles the FT(I). R strongly depends on τ_g.

Delay correction

$$R = |V| \cos(2\pi\nu_0\tau_g - \varphi_{\mathrm{V}}) \, \frac{\sin(\pi\Delta\nu\tau_g)}{\pi\Delta\nu\tau_g}$$

- τ_g varies with time because of the Earth rotation $\longrightarrow$ rapid decrease of R (1% for a path length difference of ~2 cm and $\Delta\nu = 1\mathrm{GHz}$)
- Tracking a source requires the **compensation of the geometrical delay**
- Inteferometry requires temporal coherence!

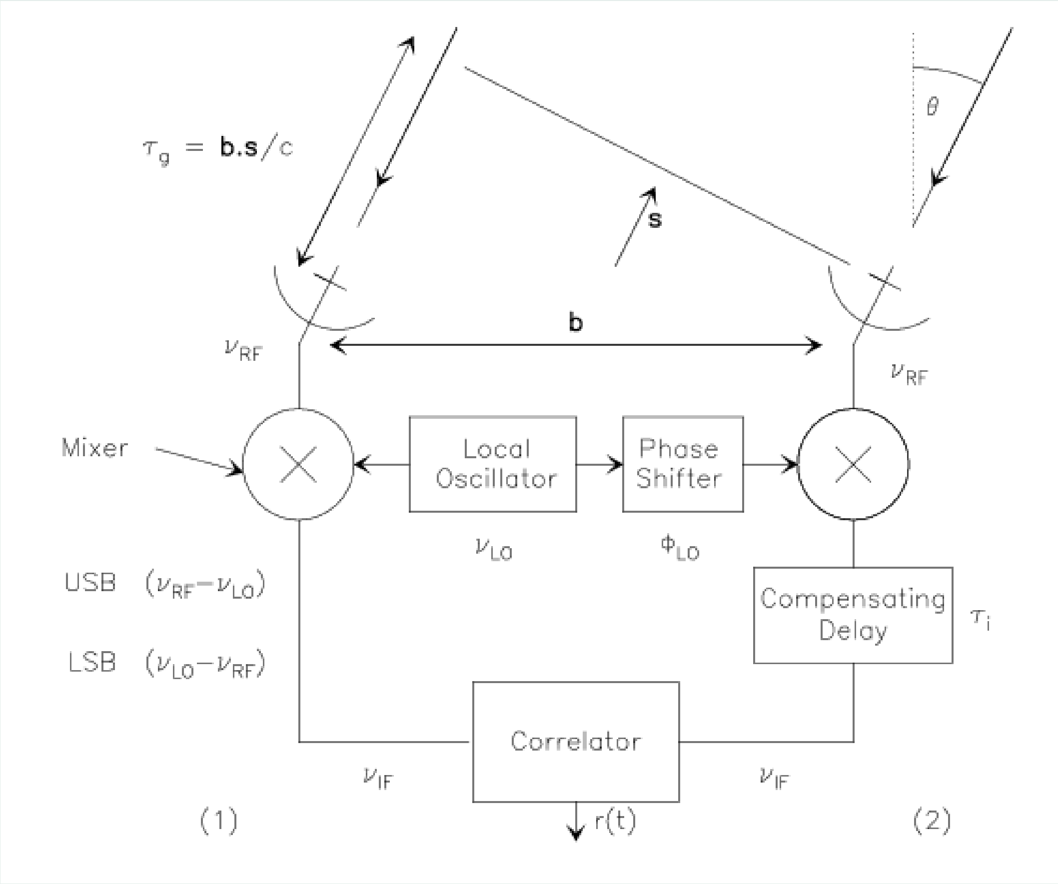
Delay correction

$$R = |V| \cos(2\pi\nu_0\tau_g - \varphi_{\mathrm{V}}) \, \frac{\sin(\pi\Delta\nu\tau_g)}{\pi\Delta\nu\tau_g}$$

- Tracking a source requires the **compensation of the geometrical delay**
- This can be achieved by introducing an **instrumental delay** in the correlator
- If delay is compensated, one can measure $R = |V| \cos(\varphi_{\mathrm{V}})$

The real interferometer

Sketch



Heterodyne detection

- In the receiver **mixer**, the incident electic field is combined with a **local oscillator** signal

$$\begin{aligned} U(t) &= E\cos\left(2\pi\nu t+\varphi\right) \\ U_{\mathrm{LO}}(t) &= E_{\mathrm{LO}}\cos\left(2\pi\nu_{\mathrm{LO}}t+\varphi_{\mathrm{LO}}\right) \\ \nu_{\mathrm{LO}} &\simeq \nu \end{aligned}$$

- The mixer is a **non-linear** element:

$$I(t) = a_0 + a_1(U+U_{\mathrm{LO}}) + a_2(U+U_{\mathrm{LO}})^2 + a_3(\ldots)^3 + \ldots$$

Heterodyne detection

- There are terms at various frequencies and harmonics
- A **filter** selects the frequencies such that;

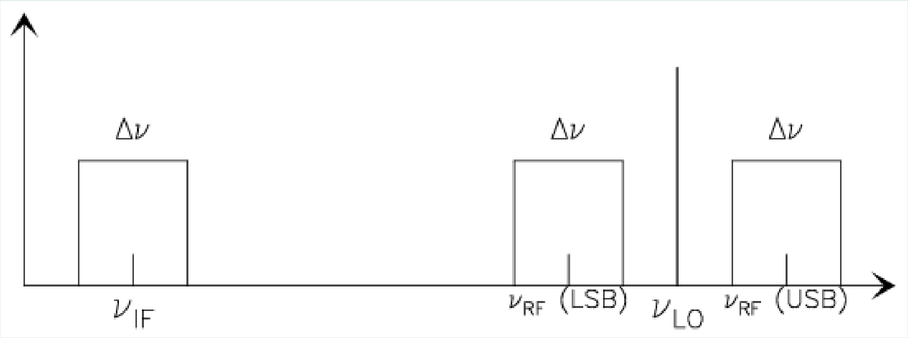
$$\nu_{\mathrm{IF}} - \Delta\nu/2 \leq |\nu - \nu_{\mathrm{LO}}| \leq \nu_{\mathrm{IF}} + \Delta\nu/2$$

- ν_{IF} is the **intermediate frequency**
- ν_{IF} such that amplifiers and transport elements available
- PdBI: ν_{IF} = 4–8 GHz, ALMA: ν_{IF} =4–8 GHz, NOEMA: ν_{IF} =4–12 GHz

Heterodyne detection

- The receiver output is

$$I(t) \propto E\,E_{\mathrm{LO}}\cos\left(\pm\left(2\pi(\nu-\nu_{\mathrm{LO}})t+\varphi-\varphi_{\mathrm{LO}}\right)\right)$$

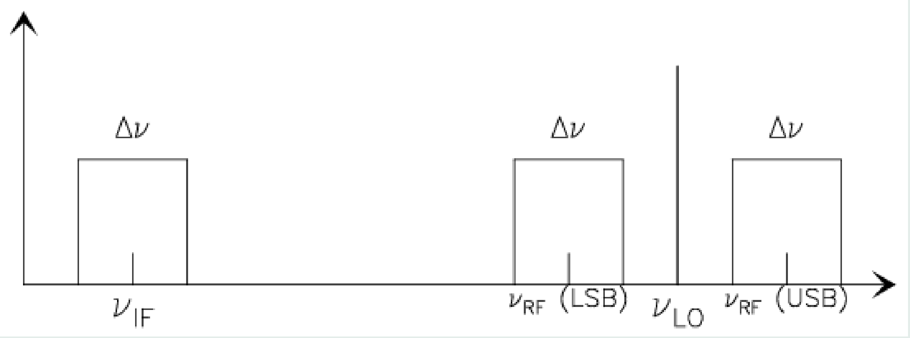


LSB **USB**

Heterodyne detection

- The receiver output is

$$I(t) \propto E\, E_{\mathrm{LO}} \cos\Big(\pm \big(2\pi(\nu - \nu_{\mathrm{LO}})t + \varphi - \varphi_{\mathrm{LO}}\big)\Big)$$

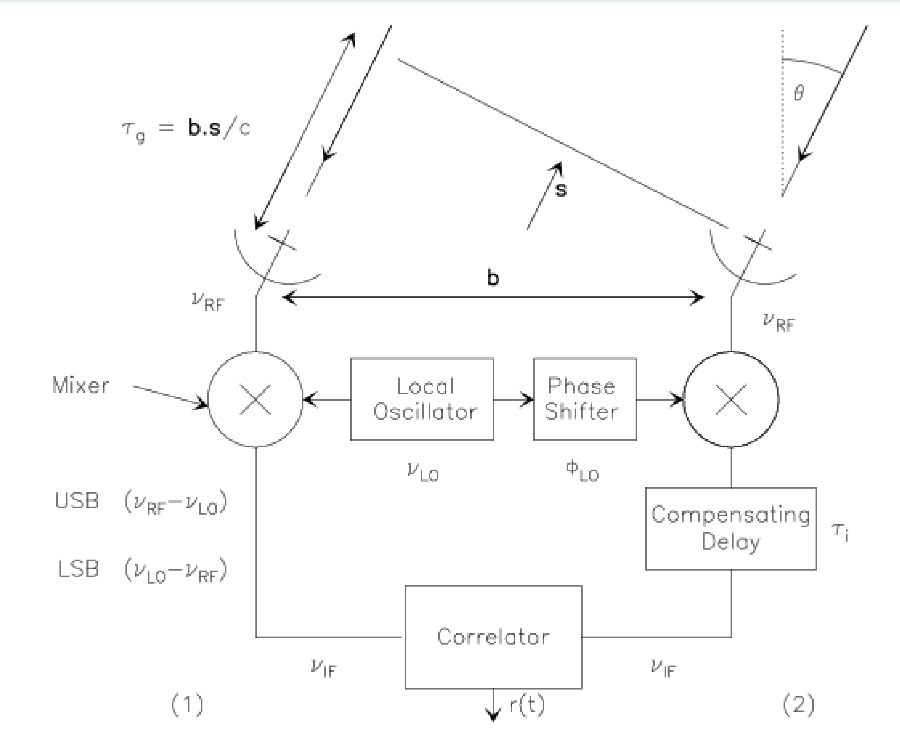


LSB **USB**

Heterodyne detection

- **DSB** receivers accept both LSB and USB frequencies, i.e. their output is the sum of LSB and USB
- **SSB** receivers accept only LSB or USB (response very strongly frequency dependant)
- **2SB** receivers are 2 DSB receivers combined such that the two bands are independently output (and processed)

Delay tracking



- A compensating delay is introduced in one of the branch of the interferometer, **on the IF signal**
- Equivalent to the delay lines in IR interferometers

The real interferometer
Delay tracking

- Phases of the two signals (USB):

$$\varphi_1 = 2\pi\nu\tau_g \qquad \varphi_1 = 2\pi\nu\tau_g = 2\pi(\nu_{\mathrm{LO}} + \nu_{\mathrm{IF}})\tau_g$$
$$\varphi_2 = 0 \qquad \varphi_2 = 2\pi\nu_{\mathrm{IF}}\tau_i$$

- Correlator output:

$$R = |V|\cos(2\pi\nu\tau_g - \varphi_{\mathrm{V}})$$
$$R = |V|\cos(\varphi_1 - \varphi_2 - \varphi_{\mathrm{V}})$$
$$R = |V|\cos(2\pi\nu_{\mathrm{LO}}\tau_g - \varphi_{\mathrm{V}})$$

Fringe Stopping

- Delay tracking not enough because applied on the IF
- Solution: in addition to delay tracking, **rotate the phase of the local oscillator** such that at any time:

$$\varphi_{\mathrm{LO}}(t) = 2\pi\nu_{\mathrm{LO}}\tau_g(t)$$

- τ_g is computed for a reference position = **phase center**
- Phase center = pointing center in practice, though not mandatory

Fringe stopping

- Phases of the two signals (USB):

$$\begin{aligned}\varphi_1 &= 2\pi\nu\tau_g \;= 2\pi(\nu_{\mathrm{LO}} + \nu_{\mathrm{IF}})\tau_g \\ \varphi_2 &= 2\pi\nu_{\mathrm{IF}}\tau_i + \varphi_{\mathrm{LO}} \\ \varphi_{\mathrm{LO}} &= 2\pi\nu_{\mathrm{LO}}\tau_g\end{aligned}$$

- Correlator output:

$$\begin{aligned}R &= |V|\cos(\varphi_1 - \varphi_2 - \varphi_{\mathrm{V}}) \\ R &= |V|\cos(-\varphi_{\mathrm{V}})\end{aligned}$$

Complex correlator

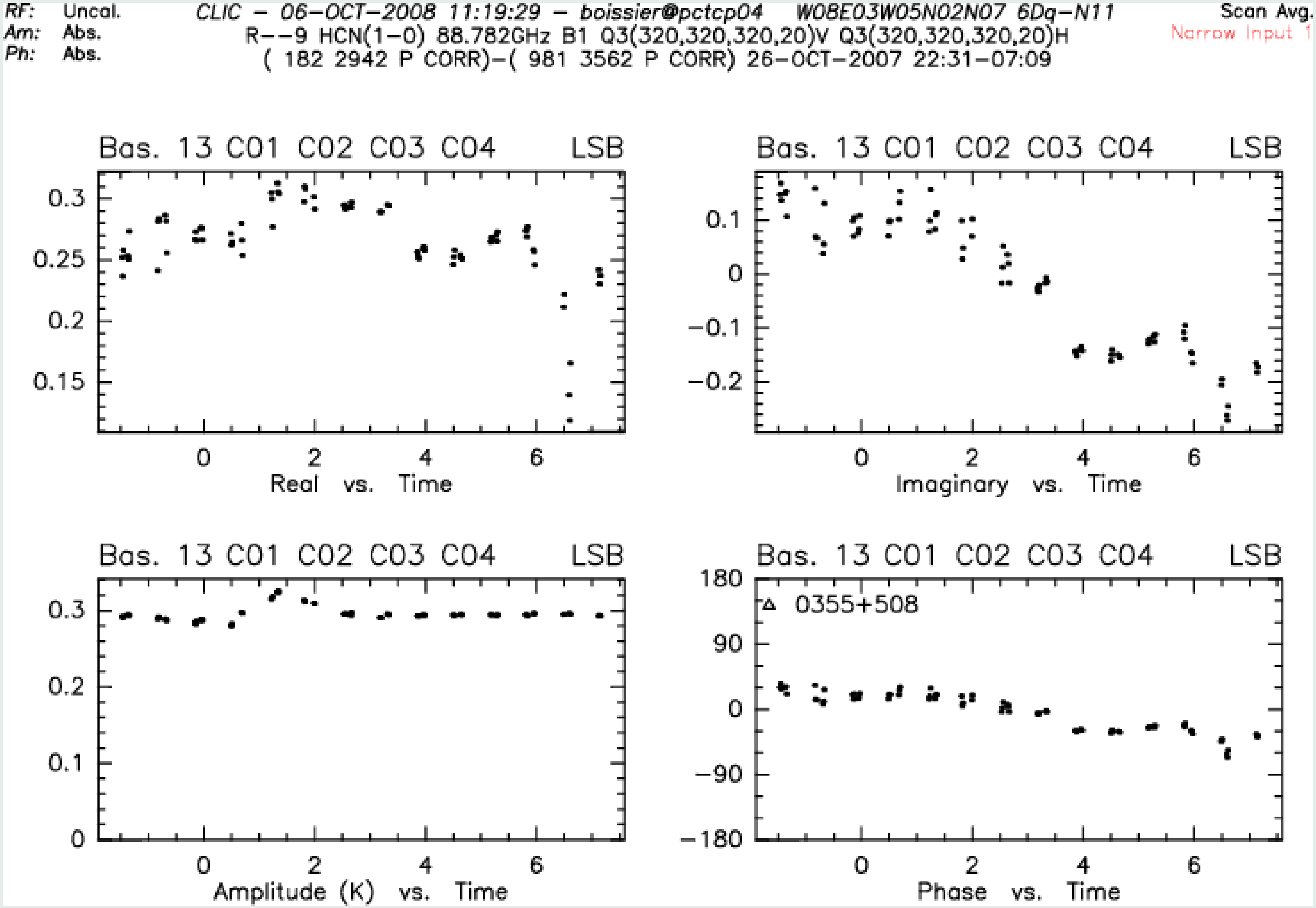
- After fringe stopping:

$$R = |V| \cos(-\varphi_{\mathrm{V}})$$

- The corrections were so good that there is **no time or delay dependance** any more $\longrightarrow$ cannot measure $|V|$ and φ_{V} separately.
- A second correlator is necessary, with one signal phase shifted by $\pi/2$: $R_i = |V| \sin(-\varphi_{\mathrm{V}})$
- **The complex correlator measures directly the visibility**

Complex correlator

- The correlator measures the real and imaginary parts of the visibility. **Amplitude and phases are computed off-line.**
- Amplitude and phases have more physical sense
  - Visibility amplitude = **correlated flux**
  - The atmosphere adds a **phase** to the incoming signals
    $\longrightarrow$ measured phase = visibility + $\varphi_1 - \varphi_2$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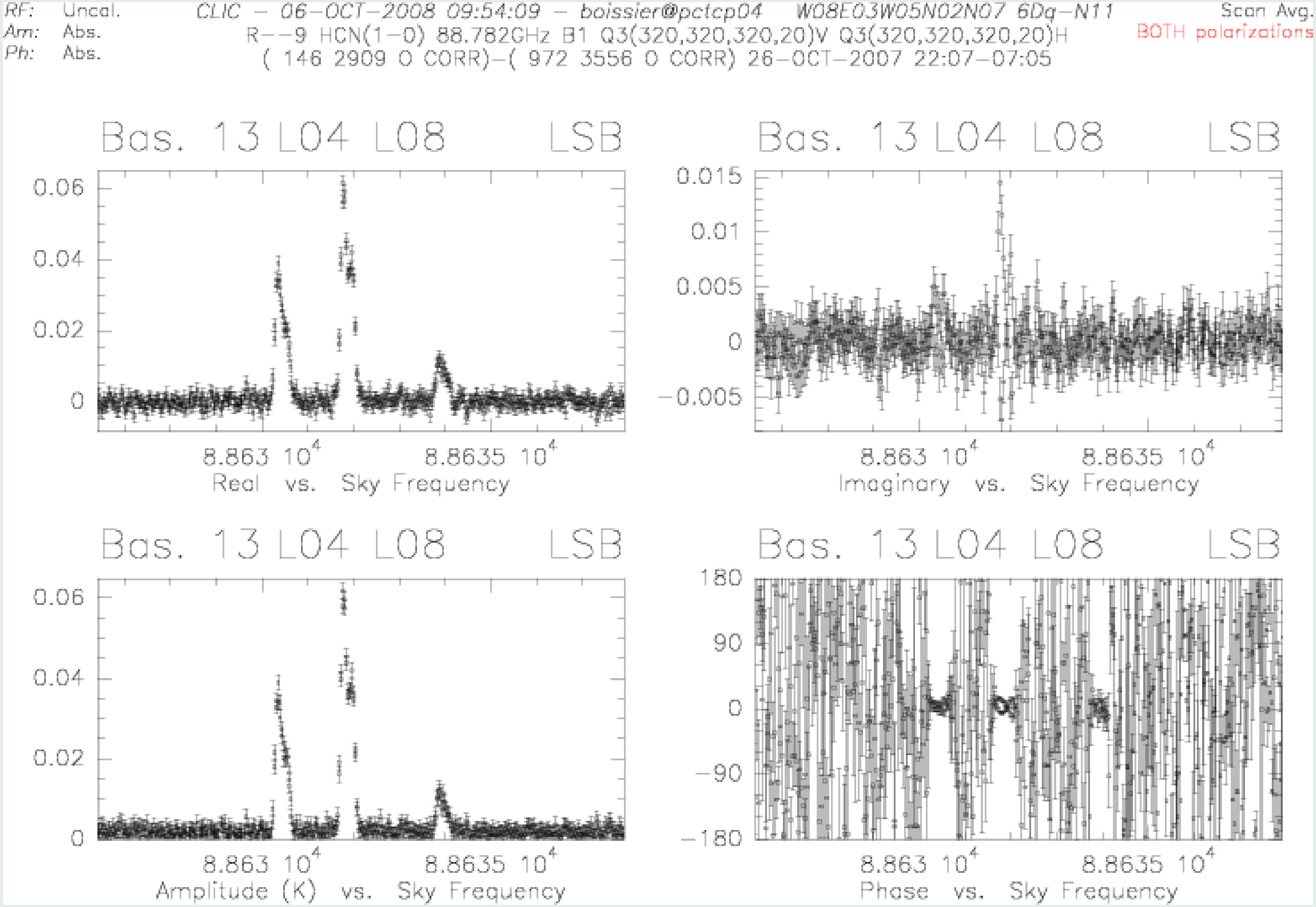
Am: Abs.
Ph: Abs.
CLIC - 06-OCT-2008 11:19:29 - boissier@pctcp04 W08E03W05N02N07 6Dq-N11
R--9 HCN(1-0) 88.782GHz B1 Q3(320,320,320,20)V Q3(320,320,320,20)H
(182 2942 P CORR)-(981 3562 P CORR) 26-OCT-2007 22:31-07:09
Scan Avg.
Narrow Input 1
Bas. 13 C01 C02 C03 C04 LSB
Real vs. Time
Bas. 13 C01 C02 C03 C04 LSB
Imaginary vs. Time
Bas. 13 C01 C02 C03 C04 LSB
Amplitude (K) vs. Time
Bas. 13 C01 C02 C03 C04 LSB
0355+508
Phase vs. Time

Spectroscopy

- Remember the Wiener-Kichnine theorem?
- Calculate the correlation function for several delay $\delta\tau \longrightarrow$ measurement of the **temporal correlation** $\longrightarrow$ FT to get the spectra:

$$V_\nu(u, v, \nu) = \int V(u, v, \tau) e^{-2i\pi\tau\nu} d\nu$$

- Nothing to do with geometrical delay compensation – $\delta\tau \sim 1/\delta\nu$ here
- Mixed up implementation in correlator software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R--9 HCN(1-0) 88.782GHz B1 Q3(320,320,320,20)V Q3(320,320,320,20)H
(146 2909 0 CORR)-(972 3556 0 CORR) 26-OCT-2007 22:07-07:05
Scan Avg.
BOTH polarizations
Bas. 13 L04 L08 LSB
Real vs. Sky Frequency
Bas. 13 L04 L08 LSB
Imaginary vs. Sky Frequency
Bas. 13 L04 L08 LSB
Amplitude (K) vs. Sky Frequency
Bas. 13 L04 L08 LSB
Phase vs. Sky Frequency

Implementing the van Cittert–Zernike theorem

1. Build a device that measures the spatial autocorrelation of the incoming signal $\longrightarrow$ **2-elements interferometer**
2. Do it for all possible scales $\longrightarrow$ **N antennas**
3. Take the FT and get an image of the brightness distribution $\longrightarrow$ **software**

Complex visibility

- Complex visibility:

$$V = |V| e^{i\varphi_{\mathrm{V}}} = \int_{Sky} A(\sigma) I(\sigma) e^{-2i\pi\nu \mathbf{b}.\sigma/c} d\Omega$$

- Going from 3-D to 2-D? ...some algrebra...
- OK providing that:

$$(\text{max. field of view})^2 \times \text{max. baseline} \ll 1$$

$$\Longrightarrow \frac{(\text{max. field of view})^2}{\text{resolution}} \ll 1$$

Complex visibility

$$V(u,v) = \int_{Sky} A(\ell,m) I(\ell,m) e^{-2i\pi\nu(u\ell+vm)} d\Omega$$

- uv plane is perpendicular to the source direction, **fixed wrt source** $\longrightarrow$ **back to Young's hole & vC-Z theorem**
- Price: limit on the field of view
- Approximation **ok in (sub)mm domain**, problem at wavelengths > cm, maybe with ALMA (long baselines, short frequencies)

(Field of view)

- Field of view is limited by
  - the **antenna primary beam**: the interferometer measures $A \times I$
  - the **2D visibility approximation**
  - the frequency averaging (bandwidth)
  - the time averaging (integration)

  $\hookrightarrow$ averaging in the uv plane; possible only if limited field of view

Aperture synthesis
(Field of view)

- Values for Plateau de Bure

θ_s	ν (GHz)	2-D Field	0.5 GHz Bandwidth	1 Min Averaging	Primary Beam
$5''$	80	$5'$	$80''$	$2'$	$60''$
$2''$	80	$3.5'$	$30''$	$45''$	$60''$
$2''$	230	$3.5'$	$1.5'$	$45''$	$24''$
$0.5''$	230	$1.7'$	$22''$	$12''$	$24''$

- Problem with 2D field: software; with bandwith: split the data for imaging; with time averaging: dump faster.
- **Primary beam is the main limit on the FOV**

Complex visibility

$$V(u,v) = \int_{Sky} A(\ell,m) I(\ell,m) e^{-2i\pi\nu(u\ell+vm)} d\Omega$$

- uv plane is perpendicular to the source direction, **fixed wrt source** $\longrightarrow$ **back to Young's hole & vC-Z theorem**
- Price: limit on the field of view
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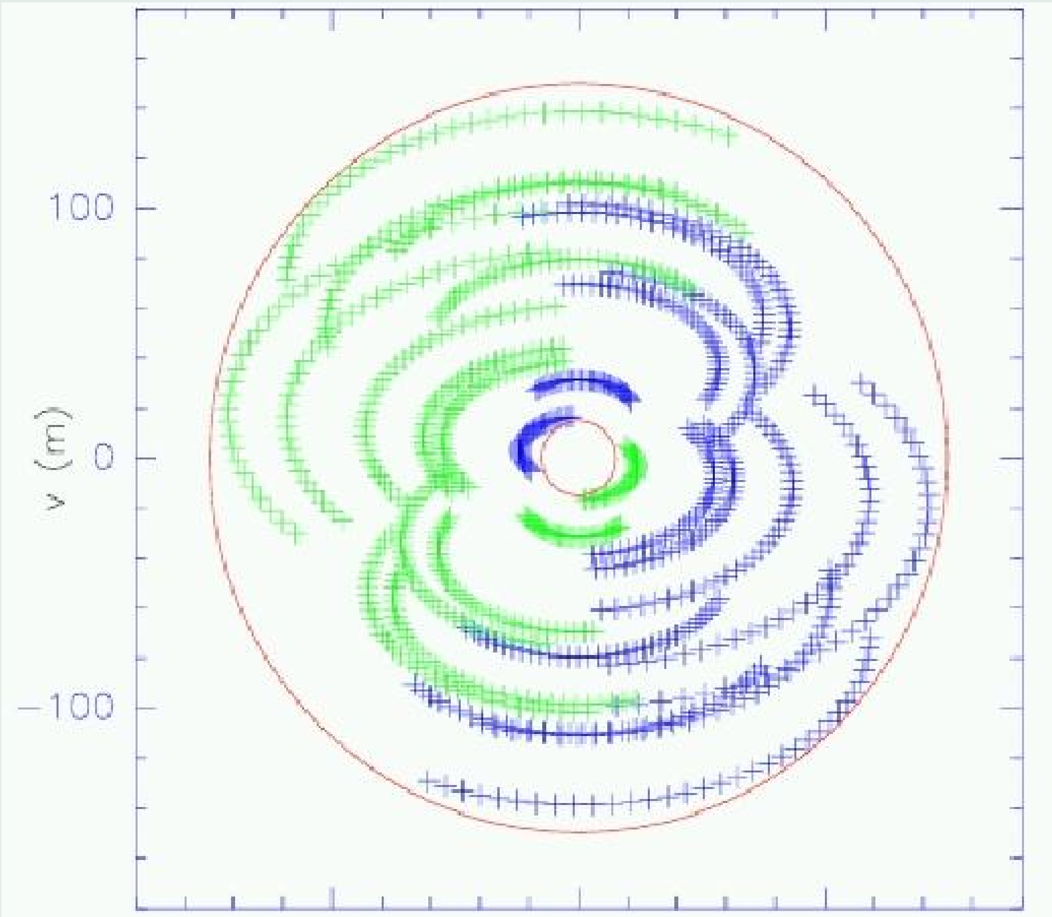
uv plane

- uv plane is perpendicular to the source direction, **fixed wrt source $\longrightarrow$ back to Young's hole**
- (u, v) is the 2–antennas **vector** baseline projected on the plane perpendicular to the source
- (u, v) are **spatial frequencies**
- ... Earth rotation ... (spherical trigonometry) ...
- (u, v) describe an **ellipse** in the uv plane (for $\delta = 0\,$deg, a line)

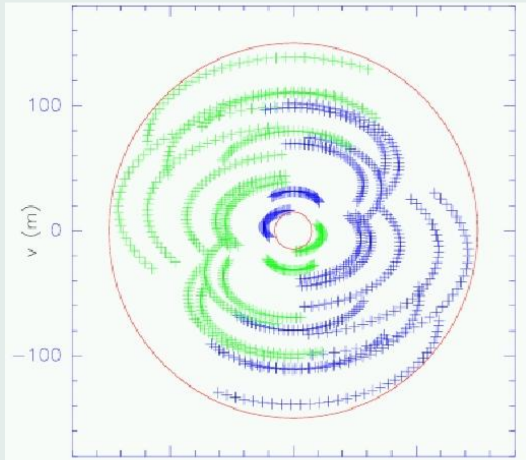
uv plane coverage



Summary

- We started with Young's hole experiment and the van Cittert–Zernike theorem
- An interferometer is **more complex**, because the two antennas (holes) are not in a plane perpendicular to the source direction $\longrightarrow$ geometrical delay, etc.
- What we are measuring is not FT(I), but the **visibility** V, which resembles a FT
- For small field of view = practical case, V is the 2D FT of the sky brighthness distribution ($\times$ the primary beam)
- **Back to the van Cittert–Zernike theorem**

Image formation

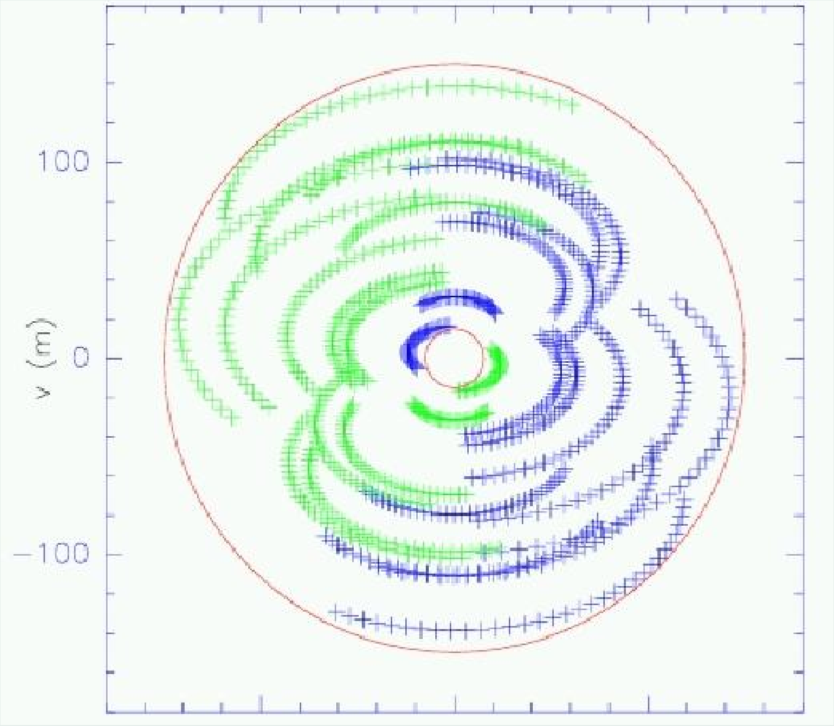


Measurements = uv plane sampling × visibilities

After FT: dirty map = dirty beam $*$ (prim. beam × sky)

The FT of the uv plane coverage gives the dirty beam = the PSF of the observations

Image formation



Max. baseline gives the angular resolution

Radiometric formula

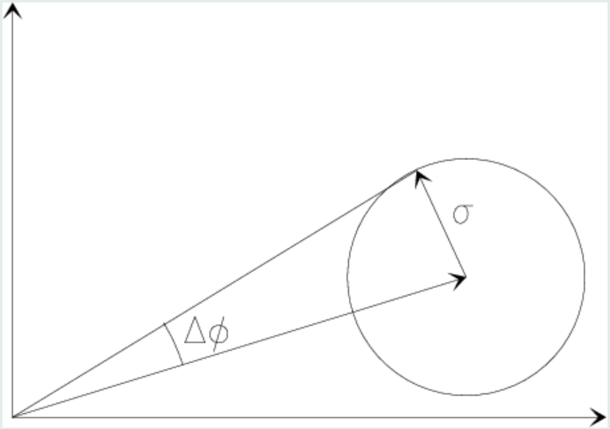
- Measurement of visibilities is limited by noise emitted by atmosphere, antenna, ground, receivers.
- The rms noise for the baseline ij is given by:

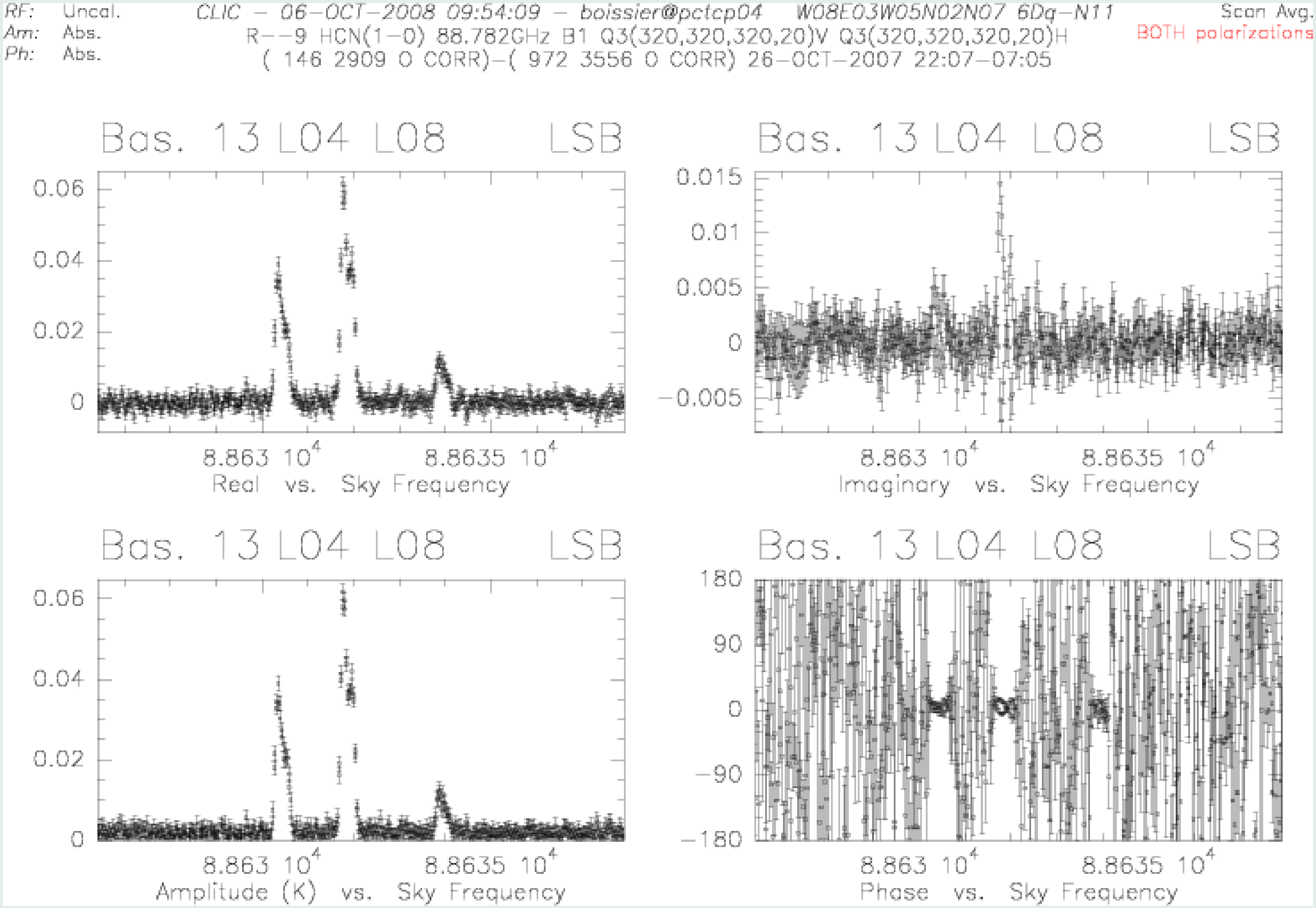
$$\delta S_{ij} = \frac{\sqrt{2}k}{A\eta_{\mathrm{A}}\eta_{\mathrm{Q}}\eta_{\mathrm{P}}} \cdot \frac{T_{\mathrm{SYS}}}{\sqrt{B\,T}}$$

  - A antenna physical aperture
  - η_{A} antenna aperture efficiency
  - η_{Q} efficiency for the correlator
  - T_{SYS} system noise temperature (single dish)
  - B bandwidth
  - T integration time
  - η_{P} phase decorrelation factor (LO jitter)

Radiometric formula

- This is the noise on the **real** and on the **imaginary** parts of the visibilities (measured independently)
- This is also the noise on the **amplitude** S
- Noise on the phase more complex, of the order of σ / S

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BOTH polarizations
Bas. 13 L04 L08 LSB
0.06
0.04
0.02
0
8.863 10^4
8.8635 10^4
Real vs. Sky Frequency
Bas. 13 L04 L08 LSB
0.015
0.01
0.005
0
-0.005
8.863 10^4
8.8635 10^4
Imaginary vs. Sky Frequency
Bas. 13 L04 L08 LSB
0.06
0.04
0.02
0
8.863 10^4
8.8635 10^4
Amplitude (K) vs. Sky Frequency
Bas. 13 L04 L08 LSB
180
90
0
-90
-180
8.863 10^4
8.8635 10^4
Phase vs. Sky Frequency

Radiometric formula

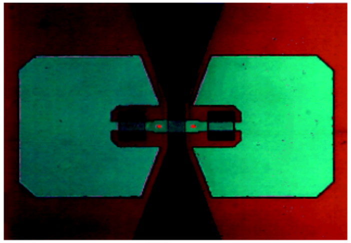
- For N identical antenna/receivers, i.e. $N(N-1)/2$ baselines, the **point-source** sensitivity is:

$$\delta S = \frac{2k}{A\eta_{\mathrm{A}}\eta_{\mathrm{Q}}\eta_{\mathrm{P}}} \cdot \frac{T_{\mathrm{SYS}}}{\sqrt{N(N-1)\,B\,T}}$$

- Scales as $\sim 1/N$
- Sensitivity to extended sources depends on angular resolution

Other instrumental issues

- Phase lock systems to control φ_{LO}
- Real-time monitoring and correction of the phase offset in the cables or fibers
- Complex phase switching is used to cancel offsets, separate/reject side bands, ...
- Antenna position measurements, to get the delay, u, v
- Antenna deformations, e.g. thermal expansion (delay)
- Accurate focus measurements (delay)
- Atmospheric phase monitoring
- ...

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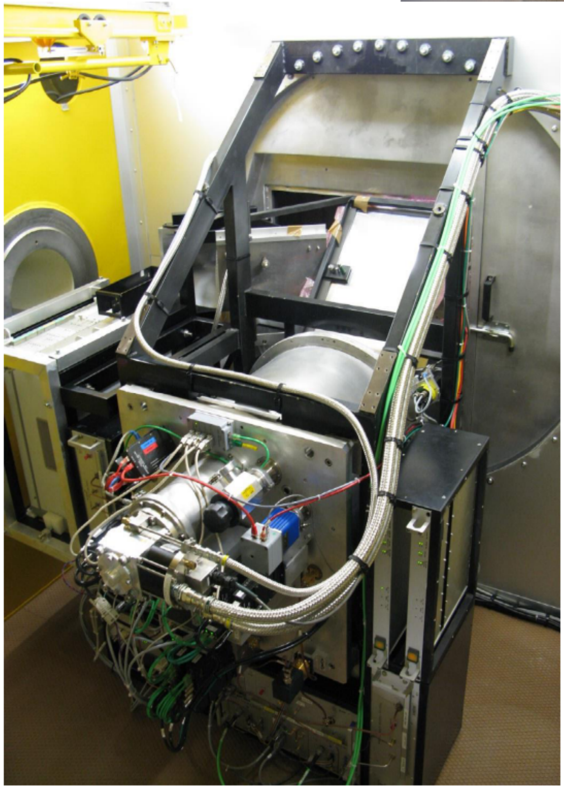


WideX 01
WideX 02
WideX 03
WideX 04