# A Calibration Scheme for the IF Polarimeter using the coherence matrix formulation

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**Description - about this document:**
A calibration scheme for the IF polarimeter at the MRT is worked out using Jones matrices and the quantum-mechanical density operator (the coherence matrix in classical theory). It is shown that the proposed calculus combines the advantages of Jones calculus and Mueller matrices.

**Related documents:**
http://iram.fr/ thum/ifpol_04.ps.gz

# Contents

# 1 Introduction

The IF polarimeter has two input channels (containing the data streams from a vertically respectively horizontally polarized receiver) and four output channels (time averages of the results of linear operations that are applied to the input data stream by the polarimeter). Thus, there are eight gains to calibrate, plus two gains describing the gains of the individual receivers with respect to the common power reference. The power reference for all gains is the mean power in the receiver pair used.

# 2 Jones Calculus and the Density Operator

The Jones calculus[1] uses $2 \times 2$ matrices with complex elements, and can handle phase information in a fully consistent way, unlike the Mueller matrices[2]. On the other hand, Jones matrices only handle pure polarization states (Mueller matrixes treat them in a fully consistent way). To treat mixed ensembles, one first has to split them into several pure ensembles, and then has to proceed by incoherent summation of the results. This is not a restriction here, if we assume that the four channels of the IF polarimeter plus the two channels bypassing it measure pure polarization states. However, this assumption does not hold anymore when the two receivers used do *not* measure merely pure polarization states. We will see below that it is nevertheless possible to extend the Jones calculus to the case of coherent addition of mixed ensembles, using the quantum-mechanical density operator. We thus have a fully consistent calculus to treat the measurement of photon ensembles in mixed polarization states with complex devices containing phase shifts and individual gains. Appendix A presents some further examples of how to use the calculus in polarimetry praxis. An effort to combine the Mueller and Jones calculus into a more powerful one capable of both treating phase information *and* mixed polarization states was already done by Hamaker et al. (A&A Supp. 117, 161, further references therein). Here I would like to show that such a calculus naturally results from the consequent application of quantum theory. Unlike the Hamaker calculus, the transformations that represent optical devices have a straightforward physical interpretation.

Jones matrices are tightly related to the Dirac description of quantum-mechanical two-state systems. The polarization state of a single photon can be either described in a base space given by horizontal and vertical polarization, or by left-hand and right-hand circular polarization. In the following, the first base space is chosen, with the $x$-axis in horizontal and the $y$-axis in vertical direction. The direction of photon propagation is taken in positive $z$-direction.

[1]Jones, R.C., 1941, J.Opt.Soc.Am. 31, 488
[2]Mueller, H., 1948, J.Opt.Soc.Am. 38, 661

A horizontally polarized photon is described by a vector

$$|x>= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1}$$

and its vertically polarized equivalent by

$$|y>= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2}$$

A photon state $|y'>$ that is linearly polarized at an angle of $45°$ with respect to the positive $x$-axis is given by

$$|y'>= \frac{1}{\sqrt{2}} (|x> + |y>) \ , \tag{3}$$

and a photon state $|x'>$ linearly polarized at an angle of $-45°$ by

$$|y'>= \frac{1}{\sqrt{2}} (|x> - |y>) \tag{4}$$

(the leading factor is to normalize probabilty to one). A circularly polarized photon can be described as a superposition of a horizontally polarized one and a $\pi/2$ phase shifted, vertically polarized one. For a right-hand circularly polarized photon state we have

$$|\mathrm{R}>= \frac{1}{\sqrt{2}} (|x> + i|y>) \ , \tag{5}$$

and for its left-hand circularly polarized counterpart

$$|\mathrm{L}>= \frac{1}{\sqrt{2}} (-|x> + i|y>) \ . \tag{6}$$

A receiver that only observes horizontally polarized light (e.g. the B 100) acts like a projection operator in quantum mechanics, which is given by the outer product $\Sigma_6$

$$\Sigma_6 = |x>< x| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot (1, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{7}$$

Similarly, the receiver measuring the vertically polarized flux (e.g. the A 100) acts like an operator $\Sigma_5$

$$\Sigma_5 = |y>< y| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot (0, 1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{8}$$

We can combine both receivers *before* time averaging to measure polarization at an angle of $45°$ from the $x$-axis, the corresponding operator being the outer product

$$\Sigma_1 = |y'>< y'| = \frac{1}{2} (|x> + |y>) \cdot (< x| + < y|) = \frac{1}{2} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \ , \tag{9}$$

respectively, for an angle of $-45°$,

$$\Sigma_2 = |x' >< x'| = \frac{1}{2}\left(|x> - |y>\right)\cdot\left(<x| - <y|\right) = \frac{1}{2}\cdot\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \quad (10)$$

Measuring right-hand circular polarization can be achieved by combining a horizontally polarized receiver with its $\pi/2$ phase-shifted, vertically polarized counterpart, i.e.

$$\Sigma_3 = |\mathrm{R} >< \mathrm{R}| = \frac{1}{2}\left(|x> + i|y>\right)\cdot\left(<x| - i<y|\right) = \frac{1}{2}\cdot\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, \quad (11)$$

and correspondingly for left-hand circular polarization

$$\Sigma_4 = |\mathrm{L} >< \mathrm{L}| = \frac{1}{2}\left(-|x> + i|y>\right)\cdot\left(-<x| - i<y|\right) = \frac{1}{2}\cdot\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \quad (12)$$

(we have to use the complex conjugates in dual space).

Thus, the measurement of these six states of polarization can be achieved by suitable coherent linear combinations of the receivers's signals, i.e.

$$\begin{aligned} \Sigma_1 \equiv A_{100} + B_{100}, \quad & \Sigma_2 \equiv A_{100} - B_{100}, \\ \Sigma_3 \equiv A^*_{100} + B_{100}, \quad & \Sigma_4 \equiv A^*_{100} - B_{100}, \\ \Sigma_5 \equiv A_{100}, \quad & \Sigma_6 \equiv B_{100}. \end{aligned} \quad (13)$$

(14)

With the time averages of these output channels, the four Stokes parameters are fully characterized (Stokes I is evidently overdetermined). In the quantum-electrodynamical expression of the density operator of a mixed ensemble of polarized photons, probabilistic concepts appear twice: first in the probability of a single photon to be in some polarization state $|p>$, and second the probability of finding this state in an ensemble of photons. Thus, the density matrix[3] reads

$$\begin{aligned} \rho &= \frac{1}{2}\left\{(\Sigma_5 + \Sigma_6) + q(\Sigma_6 - \Sigma_5) + u(\Sigma_1 - \Sigma_2) + v(\Sigma_3 - \Sigma_4)\right\} \\ &= \frac{1}{2}\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + q\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + u\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + v\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right\} \end{aligned} \quad (15)$$

(16)

since Stokes $q$ is the difference between horizontally and vertically polarized flux, Stokes $u$ the difference between flux linearly polarized at angles 45° respectively $-45°$ with respect to the horizontal direction, and finally Stokes $v$ the difference between right-hand and left-hand circularly polarized flux.

[3]Note the formal analogy of the matrices with the Pauli spin matrices.

The incoming flux made of a mixed ensemble of photons in pure polarization states is thus measured by the ensemble averages (i.e. time averages denoted [...]) of the operator that counts these photons, i.e.

$$\begin{aligned} q &= \frac{1}{2}\cdot\Big([\Sigma_6]-[\Sigma_5]\Big) = \frac{1}{2}\,\mathrm{tr}\Big(\rho\cdot(\Sigma_6-\Sigma_5)\Big) \\ u &= \frac{1}{2}\cdot\Big([\Sigma_1]-[\Sigma_2]\Big) = \frac{1}{2}\,\mathrm{tr}\Big(\rho\cdot(\Sigma_1-\Sigma_2)\Big) \\ v &= \frac{1}{2}\cdot\Big([\Sigma_3]-[\Sigma_4]\Big) = \frac{1}{2}\,\mathrm{tr}\Big(\rho\cdot(\Sigma_3-\Sigma_4)\Big) \end{aligned} \tag{17}$$

These equations explicitely show that we could work with Jones matrices, since the difference of the traces of two matrices is the trace of the difference matrix. However, this would not be true anymore if the operators $\Sigma_j$ $(j = 1, ..., 6)$ do not measure pure polarization states. In practice, we do not measure the average values $\Sigma_\mathrm{j}$, but those of a device containing individual gain factors. These average values are hereafter called $[\mathrm{S_j}]$. The Stokes parameters can be derived from these values after a gain calibration, which is detailed in the next section.

# 3 Measurement of the calibration factors

In order to calibrate the polarimeter gains, a fully linearly polarized signal from a signal generator is injected into the receivers. The polarization angle is 45°. Since small instrumental errors may introduce a deviation from this angle and thus result in a calibration error, I first follow the general case of a polarization angle $\Theta$. In the following, I assign complex gains $g_\mathrm{ij}$ to each polarimeter channel, where the first index stands for the input channel, and the second one for the polarimeter output channel. The operators of a calibration measurement, where receiver $\mathrm{B_{100}}$ is phase shifted with respect to $\mathrm{A_{100}}$ by some angle $\delta$, thus read

$$\mathrm{S}_1 = \frac{1}{2}\Big(g_{21}\mathrm{e}^{-i\delta}|x> + g_{11}|y>\Big)\cdot\Big(g_{21}^*\mathrm{e}^{i\delta}<x| + g_{11}^*<y|\Big) = \frac{1}{2}\cdot\begin{pmatrix} |g_{21}|^2 & g_{21}g_{11}^*\mathrm{e}^{-i\delta} \\ g_{21}^*g_{11}\mathrm{e}^{i\delta} & |g_{11}|^2 \end{pmatrix}, \tag{18}$$

and accordingly for the other operators

$$\mathrm{S}_2 = \frac{1}{2}\cdot\begin{pmatrix} |g_{22}|^2 & -g_{22}g_{12}^*\mathrm{e}^{-i\delta} \\ -g_{12}g_{22}^*\mathrm{e}^{i\delta} & |g_{12}|^2 \end{pmatrix}, \tag{19}$$

$$\mathrm{S}_3 = \frac{1}{2}\cdot\begin{pmatrix} |g_{23}|^2 & -ig_{23}g_{13}^*\mathrm{e}^{-i\delta} \\ ig_{13}g_{23}^*\mathrm{e}^{i\delta} & |g_{13}|^2 \end{pmatrix}, \tag{20}$$

$$\mathrm{S}_4 = \frac{1}{2}\cdot\begin{pmatrix} |g_{24}|^2 & ig_{24}g_{14}^*\mathrm{e}^{-i\delta} \\ -ig_{14}g_{24}^*\mathrm{e}^{i\delta} & |g_{14}|^2 \end{pmatrix}, \tag{21}$$

$$\mathrm{S}_5 = \begin{pmatrix} 0 & 0 \\ 0 & |g_{15}|^2 \end{pmatrix}, \tag{22}$$

and

$$S_6 = \begin{pmatrix} |g_{26}|^2 & 0 \\ 0 & 0 \end{pmatrix} . \tag{23}$$

Note how phase difference between the signal are introduced in a simple and straightforward way, making the formalism capable of treating calibration for interferometric polarization measurements. In order to calibrate the polarimeter gains, a fully linearly polarized signal from a signal generator is injected into the receivers. The polarization angle is 45°. Since small instrumental errors may introduce a deviation from this angle and thus result in a calibration error, I first follow the general case of a polarization angle $\Theta$. The density matrix for that case is given by equation 16, with $q = \cos(2\Theta)$, $u = \sin(2\Theta)$, and $v = 0$. The traces of the matrices $\rho S_j$ (with $j = 1, ..., 6$) yield the calibration curves as a function of the phase shift $\delta$:

$$\begin{aligned}
[S_1] &= \frac{1}{2} \cdot \left( |g_{21}|^2 \cos^2\Theta + |g_{11}|^2 \sin^2\Theta + \cos\Theta \sin\Theta \cdot (g_{21} g_{11}^* e^{-i\delta} + g_{11} g_{21}^* e^{i\delta}) \right) , \\
[S_2] &= \frac{1}{2} \cdot \left( |g_{22}|^2 \cos^2\Theta + |g_{12}|^2 \sin^2\Theta - \cos\Theta \sin\Theta \cdot (g_{22} g_{12}^* e^{-i\delta} + g_{12} g_{22}^* e^{i\delta}) \right) , \\
[S_3] &= \frac{1}{2} \cdot \left( |g_{23}|^2 \cos^2\Theta + |g_{13}|^2 \sin^2\Theta + i \cos\Theta \sin\Theta \cdot (-g_{23} g_{13}^* e^{-i\delta} + g_{13} g_{23}^* e^{i\delta}) \right) , \\
[S_4] &= \frac{1}{2} \cdot \left( |g_{24}|^2 \cos^2\Theta + |g_{14}|^2 \sin^2\Theta + i \cos\Theta \sin\Theta \cdot (g_{24} g_{14}^* e^{-i\delta} - g_{14} g_{24}^* e^{i\delta}) \right) , \\
[S_5] &= \sin^2\Theta |g_{15}|^2 , \\
[S_6] &= \cos^2\Theta |g_{26}|^2 .
\end{aligned} \tag{24}$$

Since all these ensemble averages are observables, they must be real. This can be easier seen by writing

$$g_{ij} = \gamma_{ij} \cdot e^{i\psi_{ij}} , \tag{25}$$

where $\gamma_{ij}$ is the amplitude of $g_{ij}$, and $\psi_{ij}$ its phase:

$$\begin{aligned}
[S_1] &= \frac{1}{2} \cdot \left( \gamma_{21}^2 \cos^2\Theta + \gamma_{11}^2 \sin^2\Theta + 2 \sin\Theta \cos\Theta \gamma_{11} \gamma_{21} \cos(\psi_{21} - \psi_{11} - \delta) \right) , \\
[S_2] &= \frac{1}{2} \cdot \left( \gamma_{22}^2 \cos^2\Theta + \gamma_{12}^2 \sin^2\Theta - 2 \sin\Theta \cos\Theta \gamma_{12} \gamma_{22} \cos(\psi_{22} - \psi_{12} - \delta) \right) , \\
[S_3] &= \frac{1}{2} \cdot \left( \gamma_{23}^2 \cos^2\Theta + \gamma_{13}^2 \sin^2\Theta + 2 \sin\Theta \cos\Theta \gamma_{13} \gamma_{23} \sin(\psi_{23} - \psi_{13} - \delta) \right) , \\
[S_4] &= \frac{1}{2} \cdot \left( \gamma_{24}^2 \cos^2\Theta + \gamma_{14}^2 \sin^2\Theta - 2 \sin\Theta \cos\Theta \gamma_{14} \gamma_{24} \sin(\psi_{24} - \psi_{14} - \delta) \right) .
\end{aligned} \tag{26}$$

The above results show the following:

- The phase factors of the complex gains merely enter as phase differences. Least-square fits to the four $[S_j]$ as a function of $\delta$ yield the gain amplitudes and phase differences and show that the latter are essentially all the same. They can therefore be calibrated out using the phase shifter for the $B_{100}$ signals. Only the amplitudes of the gain factors remain important for data reduction.

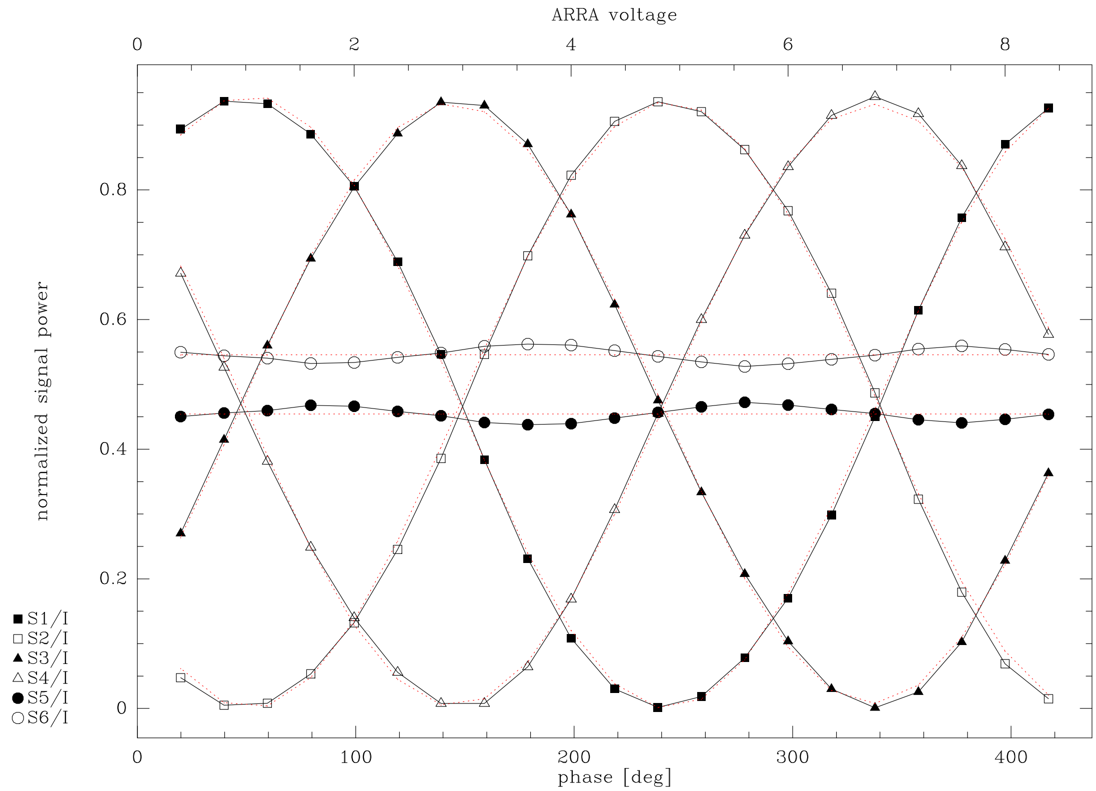


Figure 1: Example of a phase sweep measurements. The dashed red line is a least-square fit to the data.

- There is no way to determine the angle $\Theta$ from the fits, because $\gamma_{j1} \cos\Theta$ cannot be separated in the fit, neither $\gamma_{j2} \sin\Theta$. However, if the gains $g_{j1}$ are systematically, i.e. for $j = 1$ to 4, smaller or larger than the gains $g_{j2}$, an error in the experimental setup must be suspected.

- In order to obtain a good fit, the difference between the wanted phase shift delta and the true one has to be as small as possible. After a few tests, it turned out that the best fit results are obtained if the voltage-to-phase conversion factor is left as a free variable to be fitted. The result from the best fit was then used as a fixed parameter for all four sine-curve fits.

# 4 Correcting for the polarimeter gains

The application of a calibrations scheme is now straightforward: we have to obtain the average values of the operators $\Sigma_1$ to $\Sigma_6$ given by equations 7 to 12 from the measured ones $S_1$ to $S_6$ given by equations 18 to 23. The gains $g_{ij}$ (now assumed to be all real, hence equivalent to the $\gamma_{ij}$ from the previous section) are determined by means of a least square fit to a phase sweep (figure 1). It is easy to show that

$$\begin{aligned}
[\Sigma_1]-[\Sigma_2] &= \frac{[\mathrm{S}_1]}{g_{11}g_{21}}-\frac{[\mathrm{S}_2]}{g_{22}g_{12}}-\frac{[\mathrm{S}_6]}{2g_{26}^2}\cdot\left(\frac{g_{21}}{g_{11}}-\frac{g_{22}}{g_{12}}\right)-\frac{[\mathrm{S}_5]}{2g_{15}^2}\cdot\left(\frac{g_{11}}{g_{21}}-\frac{g_{12}}{g_{22}}\right),\\
[\Sigma_3]-[\Sigma_4] &= \frac{[\mathrm{S}_1]}{g_{23}g_{13}}-\frac{[\mathrm{S}_2]}{g_{14}g_{24}}-\frac{[\mathrm{S}_6]}{2g_{26}^2}\cdot\left(\frac{g_{23}}{g_{13}}-\frac{g_{24}}{g_{14}}\right)-\frac{[\mathrm{S}_5]}{2g_{15}^2}\cdot\left(\frac{g_{13}}{g_{23}}-\frac{g_{14}}{g_{24}}\right),\\
[\Sigma_6]-[\Sigma_5] &= \frac{[\mathrm{S}_6]}{g_{26}^2}-\frac{[\mathrm{S}_5]}{g_{15}^2}\,.
\end{aligned} \tag{27}$$

Inserting these average values into equation 17 yields the gain-corrected Stokes parameters[4].

It should be noted that the gains as introduced here are assumed to be the same across the whole spectral bandwidth of a polarimeter channel. However, as laboratory measurements show (S. Navarro, priv. comm.), this is not strictly the case. The above formulae may be used to seperately calibrate all the spectral channels of a polarimeter channel.

## Appendix

# A The density operator in praxis

The basic idea of the calculus used here is to not consider anymore the transfer of a pure polarization state through an arbitrary device, but to only consider the complex density operator of the mixed ensemble of polarization states we want to deal with. Thus, the effect of a device changing the polarization properties of an incoming ensemble of photons has to be described by a matrix transformation of the kind

$$\rho_{\mathrm{new}} = T^{-1}\rho_{\mathrm{old}}T\,. \tag{28}$$

Thus, optical devices in the incoming light ray are described by $2\times2$ transformation matrices $T$. Sequentially mounted devices are thus described by matrix products.

# B An example: quarter-wave plates

As an example, I demonstrate the case of the quarter-wave plate. Its effect on the incoming polarized light is that it changes Stokes $U$ into $V$, and Stokes $V$ into $-U$, leaving Stokes $Q$ and $I$ invariant. The simple Jones calculus cannot treat such a device, since the incoming light may be polarized in both Stokes $U$ *and* $V$, and thus represent a mixed ensemble. The Mueller matrix of a quarter-wave plate is easily written down,

[4] The gain correction as derived here is equivalent to that given by equations (12) to (15) in the report *A Prototype IF Polarimeter at the 30m Telescope.*

but cannot handle devices where phase shifts of the incoming signals are important. We resume the density operator equation 16, i.e.

$$\rho_{\text{old}} = \frac{1}{2} \begin{pmatrix} 1+q & u-iv \\ u+iv & 1-q \end{pmatrix} . \tag{29}$$

We look for a transformation $T$ that yields $\rho_{\text{new}}$, such that

$$\rho_{\text{new}} = T \rho_{\text{old}} T^{-1} = \frac{1}{2} \begin{pmatrix} 1+q & v+iu \\ v-iu & 1-q \end{pmatrix} . \tag{30}$$

Writing

$$T = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} , \tag{31}$$

and knowing that the inverse transformation $T^{-1}$ is given by

$$T^{-1} = \frac{1}{\det(T)} \begin{pmatrix} a_{22} & a_{21} \\ a_{12} & a_{11} \end{pmatrix} , \tag{32}$$

it is readily shown that the result is

$$T^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} , \tag{33}$$

if we take the horizontally polarized photon as phase reference. The physical interpretation is clear: the transformation $T^{-1}$ just inserts a phase lag of $\pi/2$ between the horizontally and vertically polarized photons, and thus converts linear polarization at an angle of 45° to RHC polarization.

# C Correlation polarimetry

As equation 26 show, the polarimeter output channels contain total power terms, and a correlation term (the third one). I show now that the Stokes parameters $U$ and $V$ can be directly derived from a correlation measurement, i.e.

$$R_{12}(\tau) = [S_{\text{A}}(t+\tau) S_{\text{B}}(t)] , \tag{34}$$

where the index A denotes the vertically polarized receiver, and index B the horizontally polarized one. It is of double interest here to not only measure $R_{12}(0)$: first, the Wiener-Khinchin-Theorem can be used to derive the cross-power spectral density; second, the cross-correlation is real, and can be decomposed into an even and an odd part with respect to time-reversal, yielding after a FFT the real and imaginary part of the Spectral power density.

In quantum mechanics, the correlation amplitude that some polarization state $\alpha^{(i)}$ contains an Eigenstate $|y>$ of the vertically polarized receiver (i.e. of the matrix $S_5$) and that its identical copy contains an Eigenstate $|x>$ of the horizontally polarized receiver (i.e. of the matrix $S_6$) is given by the amplitude product

$$< x|\alpha^{(i)} >< y|\alpha^{(i)} >^* = < x|\alpha^{(i)} >< \alpha^{(i)}|y > \ . \tag{35}$$

Now the probability of finding this polarization state $|\alpha^{(i)}>$ in an ensemble of photons is given by $w_{\rm i}$. The correlation then yields the backend counts $R_{12}(0)$ (assuming the ergodic hypothesis, that an ensemble average can be replaced by a time average):

$$R_{12}(0) = \sum_i w_{\rm i} < x|\alpha^{(i)} >< \alpha^{(i)}|y > \ . \tag{36}$$

Introducing identity operators and re-writing the result, this expression can be expressed in the more familiar form

$$R_{12}(0) = {\rm tr}(\rho A) \, , \tag{37}$$

where $\rho$ is the density operator of the observed mixed ensemble of polarization states, and $A$ is the outer product of the eigenvectors of the $S_5$ and $S_6$ operators, i.e.

$$A = |x> \otimes \, |y> = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \tag{38}$$

and thus, after FFT from time lag to frequency domain,

$$r_{\nu,12} = {\rm tr}\left( \frac{1}{2} \begin{pmatrix} 1+q_\nu & u_\nu - iv_\nu \\ u_\nu + iv_\nu & 1-q_\nu \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) = \frac{u_\nu + iv_\nu}{2} \, . \tag{39}$$

$U$ and $V$ are thus observed as the real and imaginary part of the cross power spectral density. Stokes $Q$ is retrieved from the difference of the autocorrelations of both receivers, as above.

We can calibrate the relative phase between the two receivers in the same way as above, i.e. the matrix $A$ now reads

$$A = \begin{pmatrix} g_{\nu,1}{\rm e}^{-i\delta} \\ 0 \end{pmatrix} \otimes (0, g^*_{\nu,2}) = \begin{pmatrix} 0 & g_{\nu,1}g^*_{\nu,2}{\rm e}^{-i\delta} \\ 0 & 0 \end{pmatrix} \, . \tag{40}$$

Thus, the output reads, taking $\rho$ with $q = \cos{(2\Theta)}$, $u = \sin{(2\Theta)}$, and $v = 0$,

$$r_{\nu,12} = \frac{1}{2} \sin{(2\Theta)} |g_{\nu,1}||g_{\nu,2}| \cdot {\rm e}^{-i(\delta + \phi_{\nu,2} - \phi_{\nu,1})} \, . \tag{41}$$

The polarimeter thus yields the uncalibrated Stokes parameters (relative to the intensity)

$$\begin{aligned} i_\nu &= \sin^2\Theta |g_{\nu,2}|^2 + \cos^2\Theta |g_{\nu,1}|^2 \, , \\ q_\nu &= \sin^2\Theta |g_{\nu,2}|^2 - \cos^2\Theta |g_{\nu,1}|^2 \, , \\ u_\nu &= 2\Re(r_{\nu,12}) = +\sin{(2\Theta)} |g_{\nu,1}||g_{\nu,2}| \cos{(\delta + \phi_{\nu,2} - \phi_{\nu,1})} \, , \\ v_\nu &= 2\Im(r_{\nu,12}) = -\sin{(2\Theta)} |g_{\nu,1}||g_{\nu,2}| \sin{(\delta + \phi_{\nu,2} - \phi_{\nu,1})} \, , \end{aligned} \tag{42}$$

where $\phi_{\nu,1}$ and $\phi_{\nu,2}$ are the phases of the complex gains $g_{\nu,1}$ and $g_{\nu,2}$. Once again, the fit cannot distinguish between a misalignment of the oscillator position and a gain imbalance. Assuming a perfect gain calibration of the two receivers, the alignment of the oscillator source may be determined from the phase sweep. The result also shows that $u_\nu$ and $v_\nu$ may be retrieved either from the real and imaginary part of the cross-power spectral density (i.e. from the even and odd part of the cross-correlation with respect to time reversal), or by inserting suitable phase shifts into the time-domain response (i.e. $\delta_1 = \phi_1 - \phi_2$ respectively $\delta_2 = \phi_1 - \phi_2 - \pi/2$ to retrieve $u$ or $v$). The latter approach is evidently only possible for continuum work, since the phase shifts are IF dependent.

For spectroscopic work, the role of the calibration is twofold:

- Determine the phase shift $\delta$ between both receivers (cf. the delay calibration in radio interferometry) at some reference IF (e.g. the IF of the oscillator signal). This fixes the sign of Stokes $V$.

- Calibrate the remaining complex bandpass response (cf. the radio-interferometric bandpass calibration) with a fully polarized continuum source of flat spectral index. Astronomical sources may be used: baseline problems cancel out, since the atmospheric contribution should be unpolarized, and hence the signals in both receivers fully uncorrelated. Whether this calibration source is polarized in $u$ or in $v$ does not matter: we can shift the power between them by adjusting the phase $\delta$.

Once the complex bandpass calibrations factors are determined, it is possible to get calibrated Stokes spectra from the complex cross-correlation product (the index $c$ means "calibrated"):

$$\begin{aligned} u_{\nu,\mathrm{c}} &= \frac{1}{|g_{\nu,1}||g_{\nu,2}|} \cdot \left(\Re(r_{\nu,12}) \cos\Delta\phi_\nu + \Im r_{\nu,12} \sin\Delta\phi_n u\right), \\ v_{\nu,\mathrm{c}} &= \frac{1}{|g_{\nu,1}||g_{\nu,2}|} \cdot \left(\Im(r_{\nu,12}) \cos\Delta\phi_\nu - \Re r_{\nu,12} \sin\Delta\phi_n u\right), \end{aligned} \tag{43}$$

where $\Delta\phi_\nu = \phi_{\nu,1} - \phi_{\nu,2}$.

# D Interferometric polarization

Polarimetry with an interferometer equipped with dual-polarization receivers works in principle as described in the previous section. It will be treated in another technical report.